

# Modular Forms as counting functions in Quantum Gravity

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CEMS Day 2026

January 27, 2026

- Quantum Gravity - understanding the microscopic emergence of spacetime geometry.
- What is the statistical mechanics underlying gravitational solutions of Einstein's equations?
- Any model of QG must provide a statistical modeling
- Analogy- Ideal gas laws from statistical mechanics of atoms.

- Solution of Einstein's equations with thermodynamic properties related to geometry.
- Specifically  $S_{BH} = \text{Horizon Area}/4 = \log d_{Stat}$  (Bekenstein- Hawking)
- Universal Formula  $\rightarrow$  excellent testing ground for QG models
- $\log(d_{stat}) = A/4 + c_1 \text{Log}A + c_2/A + c_3/A^2 + \dots$

Statistical mechanics = Mathematical rules of information organisation

- Degeneracy and Entropy invariant under symmetries.
- Given a strong enough symmetry group  $G$ , operating upon a microstate of the system by  $g \in G$  will generate another microstate- and counting all such images yields  $d = e^S$ .
- GOAL: Given parameters  $n$  that define a black hole, find a generating function for  $d(n)$  i.e. find the counting function

$$Z(x) = \sum_n d(n) e^{2\pi i x n} \quad (1)$$

such that  $d(n) \approx e^{S_{BH}}$

- Ideal Gas = Supersymmetric (BPS) black hole- Toy model
- Zero Temperature immortal systems with a special property!
- The Near Horizon geometry is  $AdS_2 \rightarrow SI(2, \mathbb{R})$  isometry
- Quantum Counting function transforms well under  $SI(2, \mathbb{Z})$

## Modular forms!

Given UHP :  $\tau = \tau_1 + i\tau_2$  with  $\tau_2 > 0$

Modular form of weight  $k$ :

$$f_k\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f_k(\tau) \quad (2)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2, \mathbf{Z}) \quad (3)$$

Jacobi form of weight  $k$  and index  $m$ :

$$f\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k e^{2\pi imc \frac{z^2}{c\tau + d}} f(\tau, z) \quad (4)$$

$$f(\tau, z + \lambda\tau) = e^{-2\pi im(\lambda^2\tau + 2\lambda z)} f(\tau, z) \quad (5)$$

$$f(\tau) = \sum_{n \geq -N} d(n) e^{2\pi i\tau n} = \sum_n d(n) q^n \quad (6)$$

$$f(\tau, z) = \sum_{n, \ell} d(\Delta = 4mn - \ell^2, \ell) q^n y^\ell \quad (7)$$

$n \geq 0 \rightarrow$  *Cuspidal*  $\rightarrow d(n) \approx n^C$

$n \geq -N \rightarrow$  weakly holomorphic  $\rightarrow d(n) \approx e^{\sqrt{n}}$

$$S_{BH} = \sqrt{n} \quad (8)$$

Therefore Weakly Holomorphic Forms can count BPS black holes!  
 Procedure- Derive the counting formula from microscopic modelling of BPS BH as bound states of strings/branes (atoms) wrapped on compact manifolds The statistical counting formula is indeed a weakly holomorphic form! Consistency: Compute  $S_{BH}$  for Black holes from Einstein's equations + Higher derivative corrections ( $R^2, R^4$ )- compare to  $d(n)$  expanded beyond leading order. Experimental verification!

$$S_{BH} = A/4 + c_1 \text{Log}A + c_2/A + c_3/A^2 + \dots \quad (9)$$

For certain classes of BPS Black holes

**We know all the coefficients!**

$$\begin{aligned}
1/2BPS \text{ in } N = 4 : \frac{1}{\eta^{24}(\tau)} &= \prod_{n \geq 1} \frac{1}{q(1 - q^n)^{24}} \\
&= \sum_{n \geq -1} p_{24}(n) q^n \quad (k = -12) \\
1/8BPS \text{ in } N = 8 : \frac{\theta_1^2(\tau, z)}{\eta^6(\tau)} &= \sum_{n, \ell} C(\Delta, \ell) q^n y^\ell \quad (k = -2)
\end{aligned}$$

For  $1/\eta^{24}$ , we have poles at  $\tau = i\infty$  and  $\mathbb{Q}$

Extract  $d(n) = p_{24}(n)$  and  $C_\ell(\Delta)$ :

- $z = \frac{4\pi\sqrt{n}}{\gamma}$

$$d(n) = \int_{\mathcal{C}} d\tau \frac{1}{\eta^{24}(\tau)} e^{-2\pi i \tau n} = 2\pi d(-1) \sum_{\gamma=1}^{\infty} \frac{1}{\gamma} n^{13/2} \text{KL}(n) I_{13}(z)$$

This is obtained via  $\int_{\mathcal{C}} = \int_{0 < \tau_1 < 1, \tau_2 = 1} = \sum_{\mathcal{O}(\frac{-\delta}{\gamma}), 0 \leq -\delta < \gamma}$  and then

mapping each rational  $\frac{-\delta}{\gamma}$  to  $i\infty$  via an  $Sl(2, \mathbb{Z})$  transform.

Similarly, for the Jacobi form,

- $z = \frac{\Pi\sqrt{\Delta}}{\gamma}$

$$C_\ell(\Delta) = 2\pi C(-1) \left(\frac{\pi}{2}\right)^{7/2} \sum_{\gamma=1}^{\infty} \frac{1}{\gamma^{9/2}} \text{KL}(\Delta, \ell) \left(\frac{2}{z}\right)^{7/2} I_{\frac{7}{2}}(z)$$

- An integer is expressed as an infinite convergent sum of functions with exactly one input -  $d(-1)$
- A pathway from functions derived from dynamics of space-time geometry to natural numbers  $\rightarrow$  Exact Path integral formulation of gravity!  
What can Gravity teach us about automorphic forms?

- For 1/4 BPS in N=4, counting function is expressed in terms of an Igusa Cusp form of weight 10
- $\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix}$  in  $Sp(4, \mathbb{Z})$   $Im\rho > 0$ ,  $Im\sigma > 0$ ,  $Im\rho Im\sigma > (Imv)^2$
- $\Phi_{10}((C\Omega + D)^{-1}(A\Omega + B)) = (Det(C\Omega + D))^{10}\Phi_{10}(\Omega)$

$$\frac{1}{\Phi_{10}(\rho, \tau, v)} = \prod_{n>0} \frac{y}{pq(1-q^n)^{20}(1-y)^2} \prod_{\substack{m > 0, \\ n \geq 0, \\ \Delta > -1}} \frac{1}{(1-p^m q^n y^\Delta)^{c(\Delta)}}$$

$$= \sum_{m,n,\ell} d(m, n, \ell) p^m q^n y^\ell \quad (10)$$

- Lots of Poles:

$$n_2(\rho\tau - v^2) - m_1\rho + n_1\tau + jv + m_2 = 0, \quad m_1 n_1 + m_2 n_2 + j^2/4 = 1/4$$

- Rademacher expansion- Identify Polar terms!
- Physics tells us  $n_2 = 0$  linear poles are not black holes → choose them as polar terms
- Triple integration gives Rademacher. Polar terms themselves given by  $p_{24}(-1)$ . Linear poles predict phase transitions!- Residues of these poles capture disappearance and appearance of new states.
- Area = Entropy → Holography - The quantum field theoretic formulation of gravity lives on 1 dimension lower. Best worked out example:  $AdS_{d+1}/CFT_d \rightarrow$  Gravity as a template to study field theories.
- AdS/CQM - So the Rademacher expansion must be able to read off the holographic Quantum mechanics. Indeed! The Heat Kernel of the quantum mechanics reproduces the modified Bessel.
- The quantum mechanics model can now be defined as a theory whose heat kernel is encoded in a modular form of a specific weight-Number theoretic axioms for quantum field theory!

● **QFT** ↔ **Number Theory** ↔ **Geometry of Gravity**

Exciting times ahead