Proof-theoretical unwinding of mathematics

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What is Proof Mining?

Unwinding of Proofs

G. Kreisel (1951) was the first to formulate the program **'unwinding of proofs'** under the general question:

What more do we know if we have proved a theorem by restricted means than if we merely know that it is true?

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By suggestion of D. Scott, "unwinding of proofs" evolved into the more appealing term "**proof mining**".

Proof Mining Today

The main idea is to analyse ineffective ordinary mathematical proofs in order to obtain additional information:

- effective bounds, algorithms (rates of convergence, of metastability, of asymptotic regularity, ...);
- results of independence of certain parameters;
- generalization of the proof by weakening of premises.

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▸ In its modern form, proof mining has been shaped largely through the pioneering work of U. Kohlenbach and collaborators since the 1990s. Key advances include shifting the focus from extracting precise witnesses to obtaining quantitative bounds, introducing abstract types, and establishing general logical metatheorems that guarantee uniform bounds from proofs in settings such as metric, hyperbolic, and normed spaces.

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▸ It has applications in nonlinear analysis, ergodic theory, fixed-point theory, convex optimization, and approximation theory.

Blackboard Part. . .

Bypassing Sequential Weak Compactness

What to analyse?

The quality of these applications seems to rest on two aspects:

1st The chosen result must be **of interest** to the community with its finitary information wanted;

2nd The extracted information must be of a **simple** nature if not new

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1st The chosen result must be **of interest** to the community with its finitary information wanted;

2nd The extracted information must be of a **simple** nature if not new

- The analysis of arithmetical comprehension (lurking in common place mathematical arguments, e.g. compactness principles) would require bounds of very high complexity.
- However, that goes against the goal of "simple information" and would not be appreciated by the general mathematician.
- In most cases the use of such comprehension principles can actually be avoided: it may happen via an 'arithmetization' of the argument, ε-weakening, or by other simplifications to the proof.

Extracting simple bounds

- In [1], a macro was developed in which certain sequential weak compactness (problematic) arguments were bypassed by instead relying on a bounded collection argument.
- Moreover, the bounded collection argument is computationally tame, in the sense that it does not contribute to an increase in the complexity of the final quantitative information.

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- Moreover, the bounded collection argument is computationally tame, in the sense that it does not contribute to an increase in the complexity of the final quantitative information.
- This perspective was applied successfully in the study of strong convergence for several Halpern-type variants of the well-known proximal point algorithm, e.g. [2,3].

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[1] F.Ferreira, L.Leuştean, P.Pinto. On the removal of weak compactness arguments in proof mining. *Advances in Mathematics*, 354:106728, 55pp, 2019.

[2] L.Leuştean, P.Pinto. Quantitative results on a Halpern-type proximal point algorithm. *Computational Optimization and Applications*, 79(1):101–125, 2021.

[3] B.Dinis, P.Pinto. Quantitative Results on the Multi-Parameters Proximal Point Algorithm. *Journal of Convex Analysis*, 28(3):729–750, 2021.

The convex feasibility problem

- Many problems in convex optimization can be stated in terms of finding a point in the intersection of a family of convex and closed sets, what is known as the **convex feasibility problem**:

$$\text{find some point } x \in \bigcap_{j \in I} C_j \tag{CFP}$$

assuming *a priori* that $\bigcap_{j \in I} C_j \neq \emptyset$, i.e. the problem is *feasible*.

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- The CFP has been the subject of much research due to its broad applicability in applied mathematics: e.g. in statistics, partial differential equations (Dirichlet problem over irregular regions), solving linear equations (Kaczmarz's method), image or signal restoration, and computed tomography.

Alternating projections

- One of the most successful and well-known techniques to iteratively approximate a solution to the CFP is the method of alternating projections (MAP).

Theorem (Halperin (1962))

Let $V_1, \cdots, V_m$ be $m \geqslant 2$ closed vector subspaces of a Hilbert space. Then, for any point $x_0 \in X$ the iteration defined by

$$x_{n+1} := P_{V_1} \cdots P_{V_n}(x_n) \tag{MAP}$$

converges strongly to $P_{\bigcap_{j=1}^m V_j}(x_0)$.

The case $m = 2$ was first established by von Neumann, and (MAP) is frequently also called von Neumann's alternating projection method.

- If the sets are just assumed to be closed and convex then...

Theorem (Bregman (1965))

Let $C_1, \cdots, C_m$ be $m \geqslant 2$ closed convex subsets of a Hilbert space such that $\bigcap_{j=1}^{m} C_j \neq \varnothing$. Then, (MAP) converges weakly to a point in the intersection.

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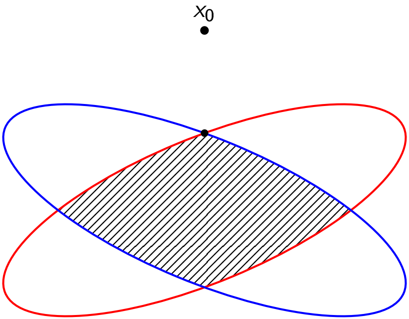

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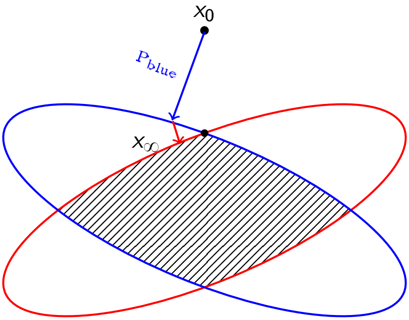

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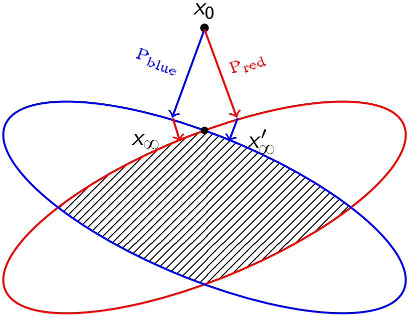

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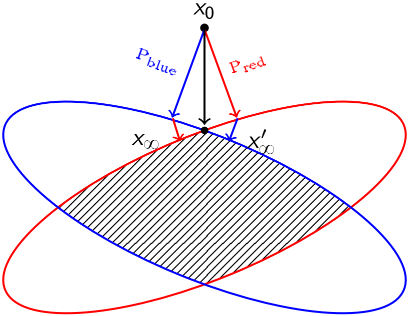

Dykstra's algorithm

Theorem (Boyle-Dykstra (1986))

Let $C_1, \cdots, C_m$ be $m \geqslant 2$ closed convex subsets of a Hilbert space such that $C := \bigcap_{j=1}^{m} C_j \neq \emptyset$. Then, for any point $x_0 \in X$ the iteration (x_n) generated by (D) converges strongly to $P_C(x_0)$.

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For $n \geqslant 1$, let C_n denote the set C_{j_n} where $j_n := [n-1]+1$ with $[r] :\equiv r \mod m$, and let P_n denote the metric projection onto C_n.

- Given an initial point $x_0 \in X$, the Dykstra's cyclic projections algorithm is defined recursively by the equations

$$\begin{cases} x_0 \in X \\ q_{-(m-1)} = \cdots = q_0 = 0 \end{cases} \quad \forall n \geqslant 1 \begin{cases} x_n := P_n(x_{n-1} + q_{n-m}) \\ q_n := x_{n-1} + q_{n-m} - x_n \end{cases} \tag{D}$$

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- When the C_i's are closed vector subspaces, the projection is a linear map and it is easy to see that the scheme (D) reduces to (MAP).

- Logical metatheorems guarantee that a proof-theoretical study of Dykstra's strong convergence result will provide us with finitary information on the Cauchy property

$$\forall \varepsilon > 0\, \exists n \in \mathbb{N}\, \forall i, j \geqslant n \left(\|x_i - x_j\| \leqslant \varepsilon\right).$$

usually in the (computationally weaker) form of a rate of <u>metastability</u>

$$\forall \varepsilon > 0\, \forall f \in \mathbb{N}^{\mathbb{N}}\, \exists n \leqslant \Phi_{b,m}(\varepsilon, f) \forall i, j \in [n; n+f(n)] \left(\|x_i - x_j\| \leqslant \varepsilon\right).$$

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$$\forall \varepsilon > 0\, \forall f \in \mathbb{N}^{\mathbb{N}}\, \exists n \leqslant \Phi_{b,m}(\varepsilon, f) \forall i, j \in [n; n+f(n)]\, (\|x_i - x_j\| \leqslant \varepsilon)\,.$$

- However, the use certain compactness arguments (in particular of weak compactness) would a priori prevent the extraction of "simple" information. Moreover, the proof structure is significantly different from previous instances where such issues were bypassed.

- The work in [4] provides a straightforward convergence proof of Dykstra's algorithm in which such troublesome arguments were bypassed, and a corresponding simple rate of metastability was extracted.

[4] P.Pinto. On the finitary content of Dykstra's cyclic projections algorithm. *Zeitschrift für Analysis und ihre Anwendungen*, 44(1/2):165–192, 2025.

- The work in [4] provides a straightforward convergence proof of Dykstra's algorithm in which such troublesome arguments were bypassed, and a corresponding simple rate of metastability was extracted.
- The theoretical justification for the elimination of such compactness arguments in the context of proof mining was provided in [5], in light of the macro developed in [1].

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[5] P.Pinto. Proof mining and the convex feasibility problem: the curious case of Dykstra's algorithm. *The Review of Symbolic Logic*, 18(3):775–811, 2025.

- In [4], it was also shown that under a metric regularity condition,

$$\forall x \in \overline{B}_r(p) \left(\bigwedge_{j=1}^{m} \|x - P_j x\| \leqslant \mu_r(\varepsilon) \to dist(x, C) \leqslant \varepsilon \right),$$

uniform rates of convergence were possible, and further that such condition was actually necessary.

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- In [6], providing a theoretical explanation to the result above, resulted in the introduction of a localized and relativized generalization of the usual concept of Fejér monotonicity. This allowed for a simple proof of the convergence in the finite dimensional setting.

[6] U.Kohlenbach, P.Pinto. Fejér monotone sequences revisited. To appear in: *Journal of Convex Analysis*, 16pp, 2025+.

Krasnoselski-Mann and the Halpern iterations

Let X be a Hilbert space, $C \subseteq X$ a nonempty closed convex subset. We say that a map $U : C \to C$ is nonexpansive if

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- Two important methods for finding fixed points of nonexpansive maps are the **Krasnoselski-Mann** and the **Halpern** iterations:

Krasnoselski-Mann's algorithm

$$\text{(KM)} \quad x_0 \in C, \quad x_{n+1} := (1 - \beta_n) U(x_n) + \beta_n x_n$$

Halpern's algorithm

$$\text{(H)} \quad x_0 \in C, \quad x_{n+1} := (1 - \alpha_n) U(x_n) + \alpha_n u$$

where $u \in C$ and $(\alpha_n), (\beta_n) \subset [0, 1]$.

- While (H) provides strong convergence under reasonable conditions, it fails to be Fejér monotone relative to the set of fixed points of U, i.e.

$$\forall p \in Fix(U)\ \forall n \in \mathbb{N}\left(\|x_{n+1} - p\| \leqslant \|x_n - p\|\right).$$

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- On the other hand, (KM) is Fejér monotone but in general only converges weakly towards a fixed point of U.

- This prompted research in the direction of modifying the algorithm (KM) in order to ensure strong convergence (changes which however seem to always prevent Fejér monotonicity).

The work in [7], introduced an iterative scheme which, taking two nonexpansive maps $U, T : C \to C$, alternates between the Halpern's and the Krasnoselski-Mann's algorithms:

Alternating Halpern-Mann

$$\text{(HM)} \quad x_0 \in C, \quad \begin{cases} y_n & := (1-\alpha_n)T(x_n) + \alpha_n u \\ x_{n+1} & := (1-\beta_n)U(y_n) + \beta_n y_n \end{cases}$$

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- The algorithm (HM) was actually studied in the general nonlinear analogue of Hilbert spaces, Hadamard spaces, in which the concept of convex combinations is still meaningful (denoted above with $\oplus$).

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- The algorithm (HM) was actually studied in the general nonlinear analogue of Hilbert spaces, Hadamard spaces, in which the concept of convex combinations is still meaningful (denoted above with $\oplus$).
- Under suitable conditions on the parameter sequences, strong convergence towards a common fixed point was established in this nonlinear setting by first providing a quantitative formulation.

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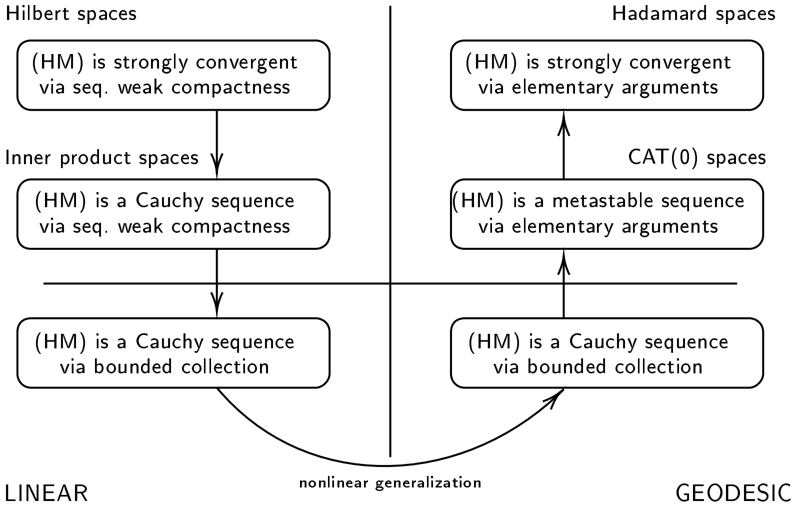
Hilbert spaces
Hadamard spaces
(HM) is strongly convergent via seq. weak compactness
(HM) is strongly convergent via elementary arguments
Inner product spaces
CAT(0) spaces
(HM) is a Cauchy sequence via seq. weak compactness
(HM) is a metastable sequence via elementary arguments
(HM) is a Cauchy sequence via bounded collection
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nonlinear generalization
LINEAR
GEODESIC

- In [8], the asymptotic regularity results of (HM) were extended to *UCW*-spaces (nonlinear gen. of uniformly convex normed spaces). Furthermore, for the (essentially) canonical choice of parameters, linear rates of asymptotic regularity, i.e. rates of convergence for

$$\lim d(x_{n+1}, x_n) = 0,$$

and quadratic rates of T- and U-asymptotic regularity, i.e. rates for

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- In [9], the results from [7,8] were extended to countable families of maps $\{T_n\}$, $\{U_n\}$ under a condition due to Nakajo, Shimoji and Takahashi (namely, the NST condition II).

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[8] L.Leuştean, P.Pinto. Rates of asymptotic regularity for the alternating Halpern-Mann iteration. *Optimization Letters*, 18:529–543, 2024.

[9] P.Firmino, P.Pinto. The alternating Halpern-Mann iteration for families of maps. Submitted, 23pp.

Nonlinear Generalizations

Proof mining and generalizations

- Proof mining provides a deeper understanding of the proof, stripping it to its essential (combinatorial) arguments.
- It has proved particularly successful in carrying results from the linear to the nonlinear setting, and a substantial portion of my research has likewise been devoted to this direction.

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Let us quickly recall certain settings of geodesic spaces:

$$\begin{aligned} \text{W-hyperbolic spaces} &\Rightarrow \text{normed spaces} \\ \text{UCW-hyperbolic spaces} &\Rightarrow \text{uniformly convex normed spaces} \\ CAT(0)\text{spaces} &\Rightarrow \text{inner product spaces} \end{aligned}$$

- Halpern's iteration (H) is known to quickly produce approximate fixed points for suitable anchoring parameters as expressed through the following rate of asymptotic regularity

$$\|x_n - Tx_n\| \leqslant \frac{2}{n+1}\|x_0 - p\|$$

in the case $\alpha_n = \frac{1}{n+2}$ (and assuming $u = x_0$) where $p \in \mathrm{Fix}(T)$ is an arbitrary fixed point of T, as recently established by Lieder (2021) and Sabach/Shtern (2017).

Moreover, this rate is known to be optimal in general.

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Moreover, this rate is known to be optimal in general.

▸ Observed by Kohlenbach, the approach of Sabach/Shtern can also employed in proof mining, and featured in the recent work of Kohlenbach, Leuștean, Cheval, Firmino, Pischke, Powell, and myself to obtain linear rates for several associated methods.

Halpern's method with adaptive parameters

In a recent work, He, Xu, Dong and Mei proposed an iteration where each anchoring parameter α_n is chosen adaptively: given a point $x_0 \in X$, define a sequence (x_n) recursively by

$$x_{n+1} := \begin{cases} x_n & \text{if } Tx_n = x_n, \\ \left(\frac{1}{\varphi_n+1}\right) x_0 + \left(\frac{\varphi_n}{\varphi_n+1}\right) Tx_n & \text{if } Tx_n \neq x_n, \end{cases}$$

where

$$\varphi_n := \frac{2\langle x_n - Tx_n, x_0 - x_n\rangle}{\|x_n - Tx_n\|^2} + 1, \quad \text{when} \quad Tx_n \neq x_n.$$

- Besides being strongly convergent, this method in particular allows for the following estimate on the asymptotic regularity

$$\|x_n - Tx_n\| \leqslant \frac{2}{\varphi_{n-1}+1}\|x_0 - p\| \leqslant \frac{2}{n+1}\|x_0 - p\|$$

for all $n \geqslant 1$, where $p \in \mathrm{Fix}(T)$ is an arbitrary fixed point of T.

- Beyond matching the speed of the optimal convergence result for Halpern's iteration due to Lieder, certain examples of application can be constructed where the above asymptotic regularity result yields substantially better theoretical estimates on the asymptotic regularity than previous ones.

- In [10], we considered the following analogous iteration in a CAT(0) space: given an arbitrary starting point $x_0 \in X$, define a sequence (x_n) via

$$x_{n+1} := \begin{cases} x_n & \text{if } Tx_n = x_n, \\ \left(\frac{1}{\varphi_n+1}\right) x_0 \oplus \left(\frac{\varphi_n}{\varphi_n+1}\right) Tx_n & \text{if } Tx_n \neq x_n, \end{cases} \qquad (\mathrm{H_{adp}})$$

where

$$\varphi_n := \frac{2\left\langle \overrightarrow{Tx_n x_n}, \overrightarrow{x_n x_0} \right\rangle}{d^2(x_n, Tx_n)} + 1, \quad \text{when} \quad Tx_n \neq x_n.$$

Here, $\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle := \frac{1}{2}\left(d^2(x,v) + d^2(y,u) - d^2(x,u) - d^2(y,v)\right)$ is the quasi-linearization function (nonlinear gen. of inner product).

[10] P.Pinto, N.Pischke. On the Halpern method with adaptive anchoring parameters. To appear in: *Mathematics of Computation*, 24pp, 2025+.

- The work in [10] begins by establishing the essential properties of the adaptive parameters.

Lemma

For all $n \in \mathbb{N}$:

1. $d^2(x_{n+1}, Tx_{n+1}) \leqslant \frac{2}{\varphi_n} \left\langle \overrightarrow{Tx_{n+1}x_{n+1}}, \overrightarrow{x_{n+1}x_0} \right\rangle$,
2. $\varphi_{n+1} > \varphi_n \geqslant n + 1$.

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2. $\varphi_{n+1} > \varphi_n \geqslant n + 1$.

- This is then used to derive the corresponding asymptotic regularity result,

$$d(x_n, Tx_n) \leqslant \frac{4}{\varphi_{n-1}} d(x_0, p) \leqslant \frac{8}{\varphi_{n-1} + 1} d(x_0, p) \leqslant \frac{8}{n+1} d(x_0, p)$$

for any $n \geqslant 1$ and $p \in Fix(T)$.

- The main convergence proof then relies on a discussion by cases (in one, elimination of weak compactness was employed; the other relies on a quantitative version of the Monotone Convergence Theorem and on simplifications of the original argument in the linear setting).

- The main convergence proof then relies on a discussion by cases (in one, elimination of weak compactness was employed; the other relies on a quantitative version of the Monotone Convergence Theorem and on simplifications of the original argument in the linear setting).
- From the quantitative study, we in particular prove a nonlinear generalization of the original convergence result.

Theorem

Let X be a Hadamard space. Let (x_n) be generated via $(\mathrm{H}_{\mathrm{adp}})$ for a given nonexpansive map T with $Fix(T) \neq \emptyset$. Then the sequence (x_n) converges to a fixed point of T.

Smooth Normed spaces

- Several results that hold in Hilbert spaces, still hold in a more general setting of (uniformly) **smooth normed spaces** where 'inner product'-like arguments are still available. In this instance, one assumes some additional conditions on the norm which are more general than assuming it to arise from an inner-product.
- Despite the relevance of these spaces, no geodesic counterpart existed in the literature.

Smooth Hyperbolic spaces

In [11], the notion of nonlinear smooth space was introduced as a space (X, d, W, π) satisfying:

(P1) $\pi(\overrightarrow{xy}, \overrightarrow{xy}) = d^2(x, y)$

(P2) $\pi(\overrightarrow{xy}, \overrightarrow{uv}) = -\pi(\overrightarrow{yx}, \overrightarrow{uv}) = -\pi(\overrightarrow{xy}, \overrightarrow{vu})$

(P3) $\pi(\overrightarrow{xy}, \overrightarrow{uv}) + \pi(\overrightarrow{yz}, \overrightarrow{uv}) = \pi(\overrightarrow{xz}, \overrightarrow{uv})$

(P4) $\pi(\overrightarrow{xy}, \overrightarrow{uv}) \leqslant d(x, y)d(u, v)$

(P5) $d^2(W(x, y, \lambda), z) \leqslant (1-\lambda)^2 d^2(x, z) + 2\lambda\pi(\overrightarrow{yz}, \overrightarrow{W(x, y, \lambda)z})$,

where $\pi : X \times X \to \mathbb{R}$ and $\overrightarrow{xy}$ denotes the pair $(x, y) \in X \times X$. The function π is a nonlinear analogue of the normalized duality map in the normed setting.

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- In [11], it was shown that this notion extends both CAT(0) spaces as well as smooth normed spaces, providing an unifying framework for several important results in functional analysis.

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- A first example of a smooth hyperbolic space which is neither a CAT(0) space nor a convex subset of a normed space was recently given in [12].

[12] P.Pinto, A.Sipoș. Products of hyperbolic spaces. Submitted, 15pp.

- A first example of a smooth hyperbolic space which is neither a CAT(0) space nor a convex subset of a normed space was recently given in [12].
- Now recall the important result due to Simeon Reich,

Theorem (Reich (1980))

Let X be a uniformly smooth Banach space, $C \subseteq X$ be a nonempty closed convex subset of X. Let $T : C \to C$ be a nonexpansive map, and $u \in C$. For each $t \in (0,1)$, consider $z_t \in C$ satisfying $z_t = (1-t)T(z_t) + tu$. If C is bounded (or $Fix(T) \neq \emptyset$) and $t \to 0$, then $(z_t)_t$ converges strongly towards a fixed point of T, namely $Q(u)$ where Q is the unique sunny nonexpansive retraction $Q : C \to Fix(T)$.

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- In [11], we established a nonlinear version of Reich's theorem.

Theorem

Let X be a complete uniformly smooth UCW hyperbolic space, C a closed nonempty bounded convex subset, and $u \in C$. Consider $T : C \to C$ a nonexpansive map. For any $t \in (0,1]$, let z_t denote the unique point in C satisfying $z_t = (1-t)T(z_t) \oplus tu$. Then, for all $(t_n) \subseteq (0,1]$ such that $\lim t_n = 0$, we have that (z_{t_n}) converges strongly towards a fixed point of T.

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- By [13], it was then possible to extend the result above for when $z_t = (1-t)T(z_t) \oplus t\phi(z_t)$, with ϕ a Meir-Keeler contraction.

[13] U.Kohlenbach, P.Pinto. Quantitative translations for viscosity approximation methods in hyperbolic spaces. *Journal of Mathematical Analysis and Applications*, 507:125823, 33pp, 2022.

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- By [13], it was then possible to extend the result above for when $z_t = (1-t)T(z_t) \oplus t\phi(z_t)$, with ϕ a Meir-Keeler contraction.
- Quantitative data for the result above was postponed for future research, but a proof mining perspective guided the remaining results of [11].

[13] U.Kohlenbach, P.Pinto. Quantitative translations for viscosity approximation methods in hyperbolic spaces. *Journal of Mathematical Analysis and Applications*, 507:125823, 33pp, 2022.

Thank you

Some additional references

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