Knot invariants, graphs, and applications in enumerative combinatorics

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Knot Invariants

- Knots - embedded circles in $\mathbb{R}^3$ (or S^3)
- Identified via isotopy
- Main knot theory question: distinguish knots

  Are two given knots isotopic?
  Is a given knot equivalent to the unknot?

- Knot invariants - invariants under isotopy, or combinatorially under Reidemeister moves

Knot Invariants - classical invariants

Classical (3d geometry) invariants:

– Fundamental group - $\pi_1(S^3 \setminus K)$

– Alexander polynomial

– Representations of the fundamental group

“Revolution” in 1980’s:

Jones polynomial, HOMFLY-PT, Reshetikhin-Turaev, quantum groups, representations...

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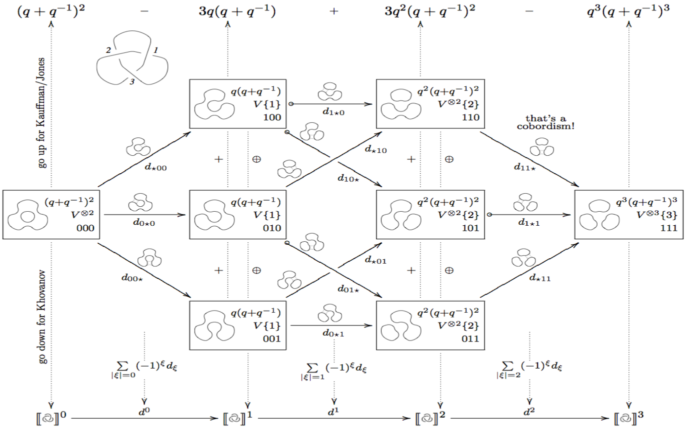
"Quantum geometry" (planar projections) vs. "Classical geometry" (3D objects)

Jones polynomial and Khovanov homology



Symmetrically colored HOMFLY–PT polynomials

Reshetikhin-Turaev invariants corresponding to $sl(n)$.
HOMFLY–PT polynomials are 2-variable invariants of knots:

$$P(K)(a, q).$$

They generalize original Jones polynomial:

$$P(a = q^2, q) = J(q).$$

There are colored versions

$$P_r(a, q), \quad r \in \mathbb{N},$$

corresponding to Sym^r representations.

LMOV conjecture

Generating function of all symmetric-colored HOMFLY-PT polynomials of a given knot K is:

$$P(x, a, q) := \sum_{r \geq 0} P_r(a, q) x^r = \exp\left(\sum_{n, r \geq 1} \frac{1}{n} f_r(a^n, q^n) x^{rn} \right),$$

$$f_r(a, q) = \sum_{i,j} \frac{N_{r,i,j} a^i q^j}{q - q^{-1}}.$$

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$N_{r,i,j}$ are BPS numbers. They represent (super)-dimensions of certain homological groups. Physically, they "count" particles of certain type (therefore are integers).

Ingredient 2: Graphs (a.k.a. Quivers)

Quivers are oriented graphs (therefore a “collection of arrows”), possibly with loops and multiple edges.

$Q_0 = \{1, \ldots, m\}$ – set of vertices.

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Let $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{N}^m$ be a dimension vector.
We are interested in moduli space of representations of Q with the dimension vector $\mathbf{d}$:

$$M_{\mathbf{d}} = \left\{ R(\alpha) : \mathbb{C}^{d_i} \to \mathbb{C}^{d_j} | \textit{for all } \alpha : i \to j \in Q_1 \right\} // G,$$

where $G = \prod_i GL(d_i, \mathbb{C})$.

C is a matrix of a quiver with m vertices.

$$P_C(x_1,\ldots,x_m) := \sum_{d_1,\ldots,d_m} \frac{(-q)^{\sum_{i,j=1}^m C_{i,j} d_i d_j}}{(q^2;q^2)_{d_1}\cdots(q^2;q^2)_{d_m}} x_1^{d_1}\cdots x_m^{d_m}.$$

q-Pochhamer symbol $(q^2;q^2)_n := \prod_{i=1}^n (1-q^{2i})$.

Quivers and motivic generating functions

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Motivic (quantum) Donaldson-Thomas invariants $\Omega_{d_1,\ldots,d_m;j}$ of a symmetric quiver Q:

$$P_C = \prod_{(d_1,\ldots,d_m)\neq 0} \prod_{j\in\mathbb{Z}} \prod_{k\geq 0} \Big(1-\big(x_1^{d_1}\cdots x_m^{d_m}\big)q^{j+2k+1}\Big)^{(-1)^{j+1}\Omega_{d_1,\ldots,d_m;j}}.$$

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Theorem (Kontsevich-Soibelman, Efimov)

$\Omega_{d_1,\ldots,d_m;j}$ *are nonnegative integers.*

Knots–quivers correspondence

[P. Kucharski, M. Reineke, P. Sulkowski, M.S., Phys. Rev. D 2017 – Editor's Suggestion, ATMP 2019]

New relationship between HOMFLY–PT / BPS invariants of knots and motivic Donaldson-Thomas invariants for quivers

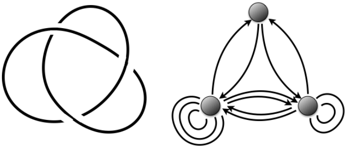

Figure: Trefoil knot and the corresponding quiver.

The generating series of HOMFLY-PT invariants of a knot matches the motivic generating series of a quiver, after setting $x_i \to x$.

BPS/LMOV invariants of knots are refined through motivic DT invariants of a corresponding quiver, and so

Theorem

For all knots for which there exists a corresponding quiver, the LMOV conjecture holds.

Consequence 1 – Some divisibilities (integrality)

If p is prime, then:

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If $r \in \mathbb{N}$, then $$r^2 \mid \sum_{d|r} \mu\left(\frac{r}{d}\right)\binom{3d-1}{d-1}.$$

$$\mu(n) = \begin{cases} (-1)^k, & n = p_1 p_2 \cdots p_k, \\ 0, & p^2 | n \end{cases}$$

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Corresponds to the fact that DT invariants are non-negative integers (in this case of the quiver of the framed unknot — one vertex, m-loop quiver)

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- Also related to other combinatorial problems – billiard partitions.

Lattice paths counting

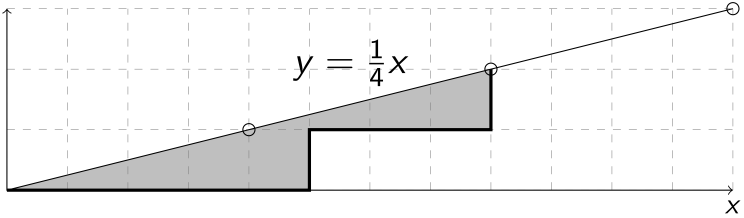

Figure: A lattice path under the line $y = \frac{1}{4}x$, and a shaded area between the path and the line.

$$y_P(x) = \sum_{k=0}^{\infty} \sum_{\pi \in \, k\text{-paths}} x^k = \sum_{k=0}^{\infty} c_k(1) x^k,$$

$$y_{qP}(x) = \sum_{k=0}^{\infty} \sum_{\pi \in \, k\text{-paths}} q^{area(\pi)} x^k = \sum_{k=0}^{\infty} c_k(q) x^k.$$

Counting lattice paths – equivalent formulation

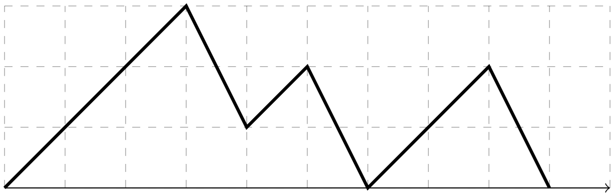

Figure: Counting of paths under the line $y = \frac{1}{2}x$ is equivalent to counting paths in the upper half plane, made of elem. steps $(1, 1)$ and $(1, -2)$.

Counting (rational) lattice paths

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Observe quotient:

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Finally, take $q \to 1$ limit ("classical" limit):

$$y(x) = \lim_{q\to 1} \frac{P(K)(q;q^2x)}{P(K)(q;x)} = 1 + \sum_{n=1}^{\infty} a_n x^n.$$

Counting (rational) lattice paths

[M. Panfil, P. Sulkowski, M.S., *Phys. Rev. D* , 2018]

Proposition

Let r and s be mutually prime.
Let $K = T_{r,s}^{f=-rs}$ be the (rs)-framed (r,s)-torus knot.
Then the corresponding coefficients a_n are equal to the number of directed lattice path from $(0,0)$ to (sn, rn) under the line $y = (r/s)x$.

Knots and quivers – results

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For example, for 2/3 slope, after computing the relevant invariants (for the bottom row) of the $T_{2,3}$ knot, and applying all the machinery, we get that the matrix of a quiver corresponding to the knot is:

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$$\begin{aligned} a_n^{(2/3)} &= \sum_{i+j=n} \frac{1}{7i+5j+1} \binom{7i+5j+1}{i} \binom{5i+5j+1}{j} \\ &= \sum_{i=0}^{n} \frac{1}{5n+i+1} \binom{5n+2i}{i} \binom{5n+1}{n-i}. \end{aligned}$$

(rediscovered Duchon formula)

Paths under the line with slope 2/3

$$a_n^{(2/3)} = \sum_{i+j=n} \frac{1}{7i+5j+1} \binom{7i+5j+1}{i} \binom{5i+5j+1}{j}.$$

$$a_n^{(2/3)}: \quad 1, 2, 23, 377, \ldots$$

j \ i	0	1	2	3
3	35	1330	37700	$\cdots$
2	5	120	2520	50375
1	1	11	152	2275
0	1	1	7	70

New results: For $2/5$ slope, i.e. $T_{2,5}$ the corresponding quiver matrix is:

$$\begin{bmatrix} 11 & 9 & 7 \\ 9 & 9 & 7 \\ 7 & 7 & 7 \end{bmatrix}.$$

$$a_n^{(2/5)} = \sum_{i+j+k=n} \frac{1}{11i+9j+7k+1} \binom{11i+9j+7k+1}{i} \binom{9i+9j+7k+1}{j} \binom{7i+7j+7k+1}{k}$$

Paths under line with slope 3/4

$$C^{(3,4)} = \begin{bmatrix} 7 & 7 & 7 & 7 & 7 \\ 7 & 9 & 8 & 9 & 9 \\ 7 & 8 & 9 & 9 & 10 \\ 7 & 9 & 9 & 11 & 11 \\ 7 & 9 & 10 & 11 & 13 \end{bmatrix} \qquad (1)$$

$$\begin{aligned} \sharp paths = & \sum_{l_1+\cdots+l_5=n} A_{(3,4)}(l_1,l_2,l_3,l_4,l_5)\times \\ & \times \frac{1}{7l_1+7l_2+7l_3+7l_4+7l_5+1}\binom{7l_1+7l_2+7l_3+7l_4+7l_5+1}{l_1}\times \\ & \times \frac{1}{7l_1+9l_2+8l_3+9l_4+9l_5+1}\binom{7l_1+9l_2+8l_3+9l_4+9l_5+1}{l_2}\times \\ & \times \frac{1}{7l_1+8l_2+9l_3+9l_4+10l_5+1}\binom{7l_1+8l_2+9l_3+9l_4+10l_5+1}{l_3}\times \\ & \times \frac{1}{7l_1+9l_2+9l_3+11l_4+11l_5+1}\binom{7l_1+9l_2+9l_3+11l_4+11l_5+1}{l_4}\times \\ & \times \frac{1}{7l_1+9l_2+10l_3+11l_4+13l_5+1}\binom{7l_1+9l_2+10l_3+11l_4+13l_5+1}{l_5}. \end{aligned} \qquad (2)$$

$$
\begin{aligned}
A_{(3,4)}(l_1,l_2,l_3,l_4,l_5)= \; & 1+28\, l_1+294\, l_1^2+1372\, l_1^3+2401\, l_1^4+33\, l_2+693\, l_1 l_2+4851\, l_1^2 l_2+11319\, l_1^3 l_2+407\, l_2^2+5698\, l_1 l_2^2+\\
&+19943\, l_1^2 l_2^2+2223\, l_2^3+15561\, l_1 l_2^3+4536\, l_2^4+34\, l_3+714\, l_1 l_3+4998\, l_1^2 l_3+11662\, l_1^3 l_3+838\, l_2 l_3+11732 l_1 l_2 l_3+\\
&+41062 l_1^2 l_2 l_3+6860 l_2^2 l_3+48020 l_1 l_2^2 l_3+18648 l_2^3 l_3+431 l_3^2+6034 l_1 l_3^2+21119 l_1^2 l_3^2+7051 l_2 l_3^2+49357 l_1 l_2 l_3^2+\\
&+28728 l_2^2 l_3^2+2414 l_3^3+16898 l_1 l_3^3+19656 l_2 l_3^3+5040 l_3^4+36 l_4+756 l_1 l_4+5292 l_1^2 l_4+12348 l_1^3 l_4+887 l_2 l_4+\\
&+12418 l_1 l_2 l_4+43463 l_1^2 l_2 l_4+7258 l_2^2 l_4+50806 l_1 l_2^2 l_4+19719 l_2^3 l_4+21294 l_3^3 l_4+482 l_4^2+6748 l_1 l_4^2+23618 l_1^2 l_4^2+\\
&+912 l_3 l_4+12768 l_1 l_3 l_4+44688 l_1^2 l_3 l_4+14914 l_2 l_3 l_4+104398 l_1 l_2 l_3 l_4+60732 l_2^2 l_3 l_4+7656 l_3^2 l_4+53592 l_1 l_3^2 l_4+62307 l_2 l_3^2 l_4+\\
&+7879 l_2 l_4^2+55153 l_1 l_2 l_4^2+32067 l_2^2 l_4^2+8086 l_3 l_4^2+56602 l_1 l_3 l_4^2+23688 l_3 l_4^3+6237 l_4^4+65772 l_2 l_3 l_4^2+37 l_5+777 l_1 l_5+\\
&+33705 l_3^2 l_4^2+2844 l_4^3+19908 l_1 l_4^3+23121 l_2 l_4^3+5439 l_1^2 l_5+12691 l_1^3 l_5+912 l_2 l_5+12768 l_1 l_2 l_5+44688 l_1^2 l_2 l_5+\\
&+7465 l_2^2 l_5+52255 l_1 l_2^2 l_5+20286 l_2^3 l_5+69524 l_2 l_3 l_5^2+35630 l_3^2 l_5^2+9010 l_4 l_5^2+63070 l_1 l_4 l_5^2+39501 l_4^2 l_5^2+25074 l_2 l_5^3+\\
&+938 l_3 l_5+13132 l_1 l_3 l_5+45962 l_1^2 l_3 l_5+15342 l_2 l_3 l_5+107394 l_1 l_2 l_3 l_5+62482 l_2^2 l_3 l_5+7877 l_3^2 l_5+3083 l_5^3+21581 l_1 l_5^3+\\
&+55139 l_1 l_3^2 l_5+64106 l_2 l_3^2 l_5+21910 l_3^3 l_5+991 l_4 l_5+13874 l_1 l_4 l_5+48559 l_1^2 l_4 l_5+16204 l_2 l_4 l_5+73269 l_2 l_4 l_5^2+75068 l_3 l_4 l_5^2+\\
&+113428 l_1 l_2 l_4 l_5+65961 l_2^2 l_4 l_5+16632 l_3 l_4 l_5+116424 l_1 l_3 l_4 l_5+135296 l_2 l_3 l_4 l_5+69335 l_3^2 l_4 l_5+25690 l_3 l_5^3+\\
&+8771 l_4^2 l_5+61397 l_1 l_4^2 l_5+71316 l_2 l_4^2 l_5+73066 l_3 l_4^2 l_5+25641 l_4^3 l_5+509 l_5^2+7126 l_1 l_5^2+27027 l_4 l_5^3+\\
&+24941 l_1^2 l_5^2+8325 l_2 l_5^2+58275 l_1 l_2 l_5^2+33894 l_2^2 l_5^2+8546 l_3 l_5^2+59822 l_1 l_3 l_5^2+6930 l_5^4 .
\end{aligned}
$$

Schröder paths

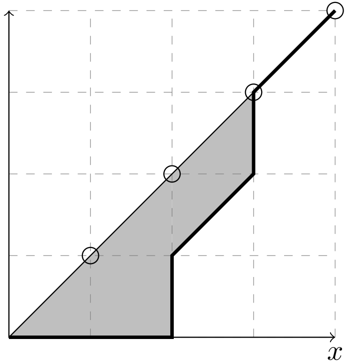

Figure: An example of a Schröder path of length 6.

Quiver corresponding to the full colored HOMFLY-PT invariants of knots in framing $f = 1$

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

This corresponds to counting paths under the diagonal line $y = x$.

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Then from the quiver generating function of C we get

$$y(x, a, q) = 1 + (q + a)x + \left(q^2 + q^4 + (2q + q^3)a + a^2\right)x^2 + \dots$$

with the height of a path measured by the power of x and the number of diagonal steps measured by the power of a.

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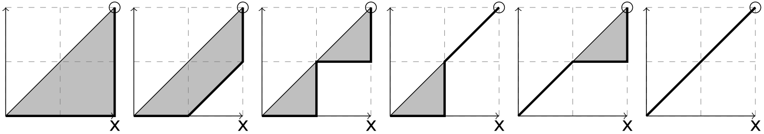


Figure: All 6 Schröder paths of height 2 represented by the quadratic term $q^2 + q^4 + (2q + q^3)a + a^2$ of the generating function.

Schröder paths and full colored HOMFLY-PT

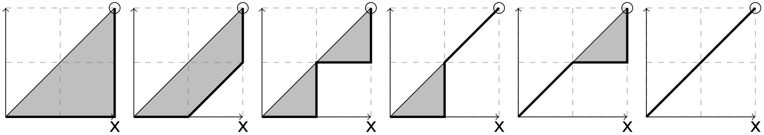
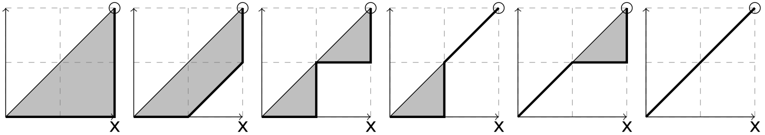


Figure: All 6 Schröder paths of height 2 represented by the quadratic term $q^2 + q^4 + (2q + q^3)a + a^2$ of the generating function.

Counting problems and full HOMFLY-PT (not just bottom row) – generalized Schröder paths under lines of rational slope [M.S., Sulkowski, *Nucl. Phys. B*, 2025]

Schröder paths and full colored HOMFLY-PT



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Counting problems and full HOMFLY-PT (not just bottom row) – generalized Schröder paths under lines of rational slope [M.S., Sulkowski, *Nucl. Phys. B*, 2025]
There are extensions, so-called generalized knots-quiver correspondence ([Ekholm, Kucharski, Longhi, 2023], [M.S., 2025]. Combinatorially, correspond to counting paths under lines not necessarily passing through origin ([D. Djordjević, M.S., 2025]).

Billiard partitions

[G. Andrews, Dragović, Radnović]
Motivated by the study of periodic trajectories of ellipsoidal billiards in d-dimensional space.

Such trajectories have $d-1$ caustics.
The set of caustics is essentially determined by a decreasing sequence of nonnegative integers (*billiard partitions*):

$$n_0 > n_1 > \ldots > n_{d-1},$$

such that

- The lowest part is even.
- Two consecutive parts cannot be both odd.

(n_0 – period of the trajectory, other n_i determine winding numbers.)

Basis billiard partitions

Every billiard partition can be written as a sum of basis billiard partition and a partition whose all parts are even.

Basis partitions are given by:

- The lowest part is 2.
- Two consecutive parts cannot be both odd.
- Consecutive parts differ by 1 or 2.

Billiard partitions and quivers

The generating series for the basis billiard partitions is given by:

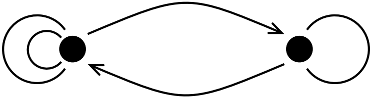
$$p_{\mathcal{B}}(a,q) = 1 + \sum_{n\geq 1, m\geq 0} p_{\mathcal{B}}(m,n)q^n = 1 + \sum_{d=1}^{\infty}\sum_{n=0}^{\infty} s(d,n),$$

where

$$s(d,2n) = a^{2n-d-1} q^{2n^2-2dn-n+d^2+2d} \begin{bmatrix} n-1 \\ 2n-d-1 \end{bmatrix}_{q^2};$$

$$s(d,2n+1) = a^{2n-d} q^{2n^2-2dn-n+d^2+3n} \begin{bmatrix} n-1 \\ 2n-d \end{bmatrix}_{q^2}.$$

Let Q be the following two-vertex quiver:



The corresponding adjacency matrix C is

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Billiard partitions and quivers

[Dragović, M.S., *Proceedings of AMS*, 2024.]
The generating series of the even basis billiard partitions can be written in the quiver form.
Moreover, the corresponding quiver is precisely the quiver Q.

Proposition

Let $P_{\mathcal{B}}^{even}(q,a,x)$ *be a generating series for even basal billiard partitions. Let* Q *be the two-vertex quiver from above. Then after setting*

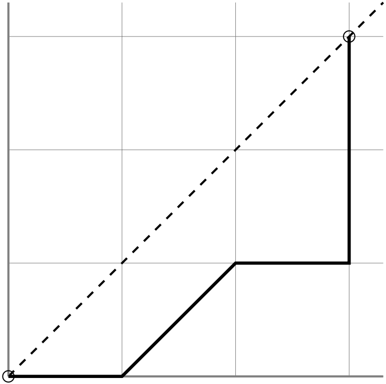
$$x_1 = q^5 x, \quad x_2 = -aq^3 x,$$

we get that the quiver generating series of Q *matches the generating series for even basal billiard partitions*

$$P_{\mathcal{B}}^{even}(q,a,x) = P_Q(x_1,x_2;q)_{|x_1=q^5x,\, x_2=-aq^3x}. \tag{3}$$

Billiard partitions and quivers

It was shown in [Panfil, Sulkowski, M.S.] that from the quiver generating series of this quiver Q, one can naturally count the Schröder paths – the lattice paths in the first quadrant below the diagonal $y = x$, so that each step can be either to the right, up, or diagonal:



Interesting fact:

– billiard partitions are natural refinement of Fibonacci numbers

– lattice paths naturally related to Catalan numbers.

Thank you for your attention!