Stability and Genericity in Dynamical Systems

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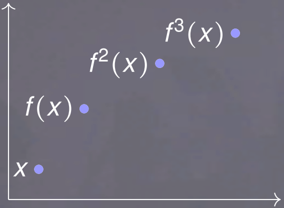
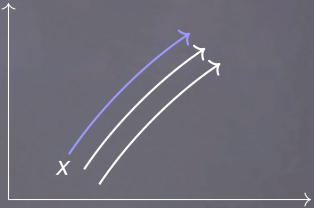
Stability and Genericity in Dynamical Systems

Discrete $f\colon M \to M$



Continuous $F\colon M \to TM$



- M compact, connected Riemannian manifold without boundary
- $f^n := \overbrace{f \circ f \circ \cdots \circ f}^{n \text{ times}},\ n \in \mathbb{Z}$
- ODE $\dot{X}^t(x) = F(X^t(x))$ has solution the flow $X^t\colon M \to M$
- X^t is a dynamical system $X^0(x) = x$ and $X^{t+s}(x) = X^t(X^s(x))$
- In the discrete case the dynamical system is f
- In the continuous case the dynamical system is generated by F via ODE

Stability and Genericity in Dynamical Systems

- First, we pick a property $\mathcal{P}$ that we'll work with (e.g. transitive, minimal, ergodic, shadowing, specification, mixing, gluing, expansiveness, . . .)
- First natural question: Does our system satisfy $\mathcal{P}$?
- Second natural question: Do 'most' dynamical systems have the property $\mathcal{P}$?
- Depending on the context, 'most' can mean open and dense, dense, or generic in the sense of Baire's 2^{nd} category (aka residual sets, dense G_δ)
- Baire's theorem implies that in a Baire space, residual sets are dense
- Ideally, we'd like the property to be open and dense, though that's usually difficult to achieve
- Dense sets are tricky to work with, since their denseness doesn't survive intersections
- Residual sets are nice to work with, since their denseness do survive intersections
- Type of mathematical statements:

  $\mathcal{P}$ is generic within a certain class of dynamical systems

Stability and Genericity in Dynamical Systems

- Once we model a phenomenon using f, a natural question arises: is the model stable?
- A diffeomorphism f is structurally stable if any C^1-nearby diffeomorphism g is topologically conjugate to f via a homeomorphism h, i.e., $h \circ f = g \circ h$

$$\begin{array}{ccc} M & \xrightarrow{\;f\;} & M \\ h\downarrow & & \downarrow h \\ M & \xrightarrow[\;g\;]{} & M \end{array}$$

- Again, we pick a property $\mathcal{P}$ that we'll work with
- Natural question: Is property $\mathcal{P}$ C^1-open?
- If not we can 'force' and consider the C^1-open set $U_{\mathcal{P}}$ of dynamical systems satisfying $\mathcal{P}$
- $U_{\mathcal{P}} \overset{?}{\Leftrightarrow}$ structural stability
- Type of mathematical statements:

$f \in U_{\mathcal{P}}$, then f is structurally stable

Perturbing Dynamical Systems

- There's a popular joke among non-dynamicists that proving results in dynamical systems is just a matter of stretch, shrink, twist here and there, . . . give it a little push, and voilà!
- In fact, they are not far from the truth
- Franks' lemma: Let $f : M \to M$ be a C^1-diffeomorphism on a compact manifold M. Then for any finite set of points $\{x_1, x_2, \dots, x_k\} \subset M$ and any linear maps

$$A_i : T_{x_i}M \to T_{f(x_i)}M$$

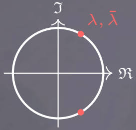
sufficiently close to $Df(x_i)$, there exists a C^1-small perturbation g of f such that

$$g(x_i) = f(x_i) \quad \text{and} \quad Dg(x_i) = A_i, \quad \text{for all } i = 1, \dots, k.$$

- In rough terms this lemma allows one to adjust the derivatives of a diffeomorphism at finitely many points while keeping the points themselves fixed
- If f is an area-preserving map on a surface, then $A_i \in SL(2, \mathbb{R})$, and g must be also area-preserving

Perturbing Dynamical Systems

- Claim: Structurally stable area-preserving maps cannot coexist with elliptic periodic points with arbitrary large period

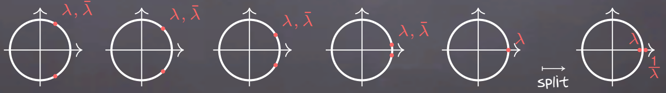


$f^n(x) = x$, $Df^n(x)$, $\sigma\left(Df^n(x)\right)$, Unit circle: $|\lambda| = 1$

- Area-preserving Franks' lemma: Let $f : M \to M$ be a C^1-area-preserving diffeomorphism on a surface M and take an elliptic periodic point x with period n. Then for any finite set of points $\{x, f(x), \ldots, f^{n-1}(x)\} \subset M$ and any linear maps $A_i : T_{f^i(x)}M \to T_{f^{i+1}(x)}M$ sufficiently close to $Df(f^i(x))$, there exists a C^1-small area-preserving perturbation g of f such that

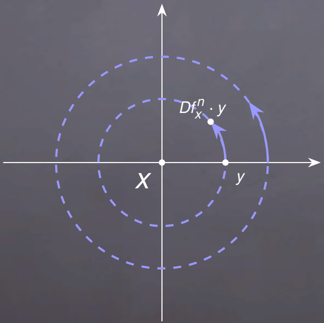
$$g(f^i(x)) = f(f^i(x)) \quad \text{and} \quad Dg(f^i(x)) = A_i, \quad \text{for all } i = 0, \ldots, n-1.$$

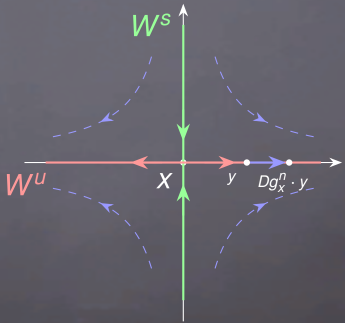
- $$Dg^n(x) = A_{n-1} \circ A_{n-2} \circ \cdots \circ A_1 \circ A_0 = \overbrace{Df(f^{n-1}(x))R_\alpha}^{A_{n-1}} \circ \overbrace{Df(f^{n-2}(x))R_\alpha}^{A_{n-2}} \circ \cdots \circ \overbrace{Df(f(x))R_\alpha}^{A_1} \circ \overbrace{Df(x)R_\alpha}^{A_0}$$



Perturbing Dynamical Systems

- Claim: Structurally stable area-preserving maps cannot coexist with elliptic periodic points with arbitrary large period
- By a C^1-small perturbation of f, we obtained a map g for which the elliptic point x became hyperbolic, thus cannot be structurally stable
- In fact the linearized dynamics is substantially different (rotation versus saddle)





Perturbing Dynamical Systems

- Question: But ... I would say that Franks' Lemma seems rather trivial, doesn't it?
- Pujals+Sambarino [Annals Math. 2009] There exist a C^2 diffeomorphism $f : M \to M$ and a neighborhood $\mathcal{U}$ of f in the C^2 topology such that f has a sequence of periodic points p_n, with unbounded periods and one normalized eigenvalue of p_n converging to 1 and any periodic point of $g \in \mathcal{U}$ of period greater than 2 is hyperbolic.

Perturbing Dynamical Systems

- Question: How about Franks' Lemma for flows?
- Morales+Pacifico+Pujals [Annals Math. 2004]

LEMMA 3.1. *Given* $\varepsilon_0 > 0$, $Y \in \mathcal{X}^2(M)$, *an orbit segment* $Y_{[a,b]}(p)$, *a neighborhood* U *of* $Y_{[a,b]}(p)$ *and a parametrized family of invertible linear maps* $A_t : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $t \in [a,b]$, C^2 *with respect to the parameter* t, *such that*

a) $A_0 = \mathrm{Id}$ *and* $A_t(Y(Y_s(q))) = Y(Y_{t+s}(q))$,

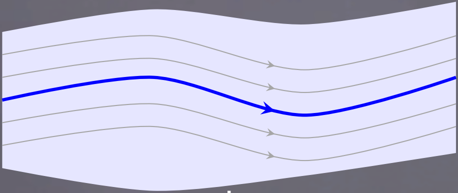
b) $\|\partial_s A_{t+s} A_t^{-1}|_{s=0} - DY(Y_t(p))\| < \varepsilon$, *with* $\varepsilon < \varepsilon_0$,

then there is $Z \in \mathcal{U}$, $Z \in \mathcal{X}^1(M)$ *such that* $\|Y - Z\| \leq \varepsilon$, Z *coincides with* Y *in* $M \setminus U$, $Z_s(p) = Y_s(p)$ *for every* $s \in [a,b]$, *and* $DZ_t(p) = A_t$ *for every* $t \in [a,b]$.

- Question: But if $Y \in \mathcal{X}^2_\mu$ is divergence-free, can we get $Z \in \mathcal{X}^2_\mu$? (μ =Lebesgue)
- Answer: Yes! B.+Rocha [JDE 2008]

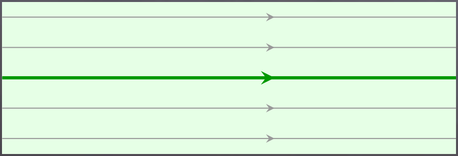
Perturbation Techniques in Adapted Coordinates

$X^t: M \to M,\ \nabla \cdot X = 0$

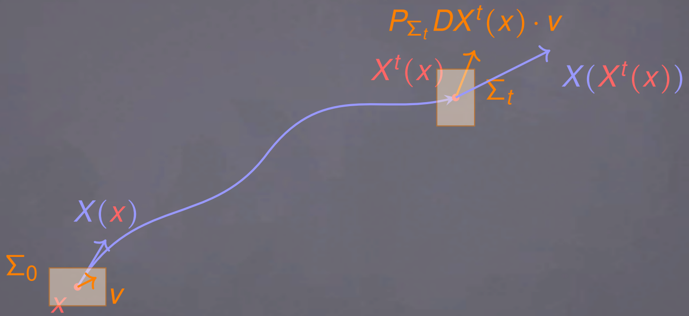


↓ Volume-preserving diffeomorphism – $\underbrace{\text{B. 2007}}_{\text{CEMS}}$, $\underbrace{\text{Cristian Barbarosie 2011}}_{\text{CEMS}}$

Trivialized flow (rectified flow)



Linear Poincaré flow



- $\Sigma_t = X(X^t(x))^\perp$
- $P_X^t(x) = \overbrace{P_{\Sigma_t}}^{\text{projection in the normal fiber bundle}} DX^t(x) \cdot v$
- $X \overbrace{\to}^{\text{integrate}} X^t \overbrace{\to}^{\text{tangent flow}} DX^t \overbrace{\to}^{\text{project}} P_X^t$

Hamiltonian flows

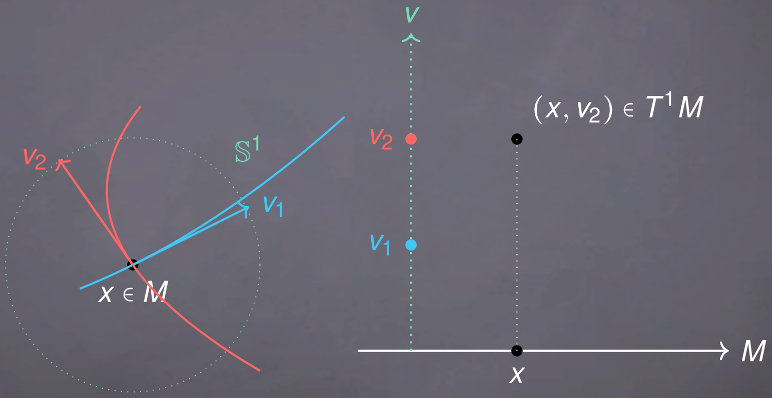
- Hamiltonian $H: M \to \mathbb{R}$ and a symplectic form ω ($\dim(M) = 2n$)
- Closed ($d\omega = 0$), nondegenerate antisymmetric bilinear 2-form
- Hamiltonian vector field $X_H: M \to TM$ is defined by $\omega(X_H(x), v) = dH_x(v), \quad \forall\, v \in T_x M$
- Hamiltonian flow $X_H^t: M \to M$
- Hamiltonian tangent flow $DX_H^t: TM \to TM$. Identify DX_H^t with a matrix in $Sp(2n, \mathbb{R})$
- Restrict the dynamics to a energy hypersurface $\mathcal{E} = H^{-1}(\{e\})$ ($\dim \mathcal{E} = 2n - 1$)
- Hamiltonian linear Poincaré flow $P_H^t: \mathcal{E}/X_H \to \mathcal{E}/X_H$. Identify P_H^t with a matrix in $Sp(2n-2, \mathbb{R})$
- $H \overset{\text{use } \omega}{\overbrace{\to}} X_H \overset{\text{integrate}}{\overbrace{\to}} X_H^t \overset{\text{tangent flow}}{\overbrace{\to}} DX_H^t \overset{\text{project}}{\overbrace{\to}} P_H^t$
- Notice we use C^2 topology on the hamiltonian H

Famous Hamiltonian flows

- Mechanical Hamiltonians are of the form $H(q,p) = K(p) + V(q)$, where K is the kinetic energy and V the potential energy
- The geodesic flow on a Riemannian manifold (M, g) can be interpreted as the motion of a free particle whose Hamiltonian is purely kinetic $H(q,p) = \frac{1}{2} g^{ij}(q)\, p_i p_j$,
- The geodesic flow for a Lorentzian metric (particular case of semi-Riemannian metrics)
- The geodesic flow for a Randers metric (particular case of Finsler metrics) defined by

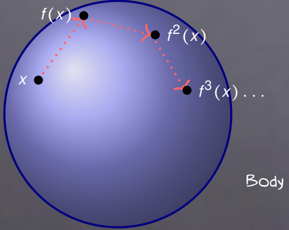
$$F(x,v) = \overbrace{\alpha(x,v)}^{\text{riemannian part}} + \overbrace{\beta(x,v)}^{\text{drift}}$$

Challenges in perturbing geodesic flows



Franks' lemma ... once again!

- Question: How about Franks' Lemma for geodesic flows?
- Contreras [Annals Math. 2010]
- Question: How about Franks' Lemma for geodesic flows (semi-riemannian)?
- Answer: Yes! B.+Lopes-Dias+$\overbrace{\text{Matias}}^{\text{CEMS}}$+Torres [PAMS 2025]
- Question: How about Franks' Lemma for flat geodesic flows on bodies?



- Answer: Yes! B.+Lopes-Dias+Gaivão+Del Magno+Torres [Adv. Math. 2024]

Linear discrete-time cocycles

- Let's describe the abstract context of linear cocycles in discrete time.
- BASE OF THE COCYCLE: Take an automorphism $f\colon M \to M$ preserving a measure μ.
- FIBER OF THE COCYCLE: Take $A\colon M \to \mathbb{R}^{n\times n}$ with some regularity L^p, C^0, Hölder, C^r, ...
- COCYCLE DYNAMICS: $\mathcal{A}\colon M \times \mathbb{R}^n \to M \times \mathbb{R}^n$ is defined by the skew-product:

$$\mathcal{A}(x, v) = (f(x), A(x)v).$$

- with iterations defined by:

$$\mathcal{A}^n(x, v) = (f^n(x), A^n(x)v),$$

$A^0(x)v = v$ for all $v \in \mathbb{R}^d$,

$$A^n(x) = A(f^{n-1}(x)) \ldots A(f(x))A(x) \quad \text{for} \quad n > 0$$

and

$$A^n(x) = A(f^n(x))^{-1} \ldots A(f^{-1}(x))^{-1} \quad \text{for} \quad n < 0.$$

Linear discrete-time cocycles

$$\begin{array}{ccccccc} M \times \mathbb{R}^n = TM & \xrightarrow{\mathcal{A}} & M \times \mathbb{R}^n & \xrightarrow{\mathcal{A}} & M \times \mathbb{R}^n & \xrightarrow{\mathcal{A}} & M \times \mathbb{R}^n \dots \\ \downarrow \pi & & \downarrow \pi & & \downarrow \pi & & \\ M & \xrightarrow{f} & M & \xrightarrow{f} & M & \xrightarrow{f} & M \dots \end{array}$$

- The most natural example is to consider a diffeomorphism $f\colon \mathbb{T}^n \to \mathbb{T}^n$ and $A = Df$ (jacobian).

$$\begin{array}{ccccccc} \mathbb{T}^n \times \mathbb{R}^n & \xrightarrow{Df} & \mathbb{T}^n \times \mathbb{R}^n & \xrightarrow{Df} & \mathbb{T}^n \times \mathbb{R}^n & \xrightarrow{Df} & \mathbb{T}^n \times \mathbb{R}^n \dots \\ \downarrow \pi & & \downarrow \pi & & \downarrow \pi & & \\ \mathbb{T}^n & \xrightarrow{f} & \mathbb{T}^n & \xrightarrow{f} & \mathbb{T}^n & \xrightarrow{f} & \mathbb{T}^n \dots \end{array}$$

- $A^n(x) = A(f^{n-1}(x)) \dots A(f(x))A(x)$ Chain rule $Df^n(x) = Df(f^{n-1}(x)) \dots Df(f(x))Df(x)$

Franks enter in the room

$$\begin{array}{ccccccc}
\mathbb{T}^n \times \mathbb{R}^n & \xrightarrow{Df} & \mathbb{T}^n \times \mathbb{R}^n & \xrightarrow{Df} & \mathbb{T}^n \times \mathbb{R}^n & \xrightarrow{Df} & \mathbb{T}^n \times \mathbb{R}^n \ldots \\
\downarrow \pi & & \downarrow \pi & & \downarrow \pi & & \\
\mathbb{T}^n & \xrightarrow{f} & \mathbb{T}^n & \xrightarrow{f} & \mathbb{T}^n & \xrightarrow{f} & \mathbb{T}^n \ldots
\end{array}$$

- $$\begin{array}{ccccccc}
\mathbb{T}^n \times \mathbb{R}^n & \xrightarrow{A_1} & \mathbb{T}^n \times \mathbb{R}^n & \xrightarrow{A_2} & \mathbb{T}^n \times \mathbb{R}^n & \xrightarrow{A_3} & \mathbb{T}^n \times \mathbb{R}^n \ldots \\
\downarrow \pi & & \downarrow \pi & & \downarrow \pi & & \\
\mathbb{T}^n & \xrightarrow{f} & \mathbb{T}^n & \xrightarrow{f} & \mathbb{T}^n & \xrightarrow{f} & \mathbb{T}^n \ldots
\end{array}$$

- $$\begin{array}{ccccccc}
\mathbb{T}^n \times \mathbb{R}^n & \xrightarrow{Dg(x)=A_1} & \mathbb{T}^n \times \mathbb{R}^n & \xrightarrow{Dg(g(x))=A_2} & \mathbb{T}^n \times \mathbb{R}^n & \xrightarrow{Dg(g^2(x))=A_3} & \mathbb{T}^n \times \mathbb{R}^n \ldots \\
\downarrow \pi & & \downarrow \pi & & \downarrow \pi & & \\
\mathbb{T}^n & \xrightarrow{g} & \mathbb{T}^n & \xrightarrow{g} & \mathbb{T}^n & \xrightarrow{g} & \mathbb{T}^n \ldots
\end{array}$$

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Lyapunov exponents

- The spectral theory of a linear map A is well-known
- The spectral theory of compositions like $A^n = A_{n-1} \circ A_{n-2} \circ \cdots \circ A_1 \circ A_0$ when $n \to \infty$ a very challenging problem
- $\lim_{n \to +\infty} \frac{1}{n} \log \|A^n\|$
- Under mild conditions on f (μ-invariant) and on A ($\log \|A\| \in L^1$) the limit

$$\lim_{n \to +\infty} \frac{1}{n} \log \|A^n(x)\| =: \lambda(A, f, x)$$

exists for μ-a.e. x

- $\underbrace{\text{Duarte}}_{\text{CEMS}}$ $\longrightarrow$ Towards stability/continuity of Lyapunov exponents $A \mapsto \lambda(A)$
- $\underbrace{\text{B.}}_{\text{CEMS}}$ $\longrightarrow$ Lyapunov exponents genericity (discontinuity)

Annals of Mathematics, **161** (2005), 1423–1485

The Lyapunov exponents of generic volume-preserving and symplectic maps

By Jairo Bochi and Marcelo Viana*

- C^1-generically chaotic behavior is rare
- Question: How about this result for flows? Answer: Yes! B. (Lopes-Dias, Rocha)

END