Random Product of 2-D matrices

P. Duarte[1] M. Durães[1] T. Graxinha[1] S. Klein[2]

[1]FCUL
Faculdade de Ciências da Universidade de Lisboa

[2]PUC-Rio
Pontifícia Universidade Católica do Rio de Janeiro

FCUL, January 2026

Probabilistic viewpoint

We study the multiplicative analogue of sums of i.i.d. random variables.

Consider a semigroup $G \subset \mathsf{Mat}_d(\mathbb{R})$. Common choices for G are $\mathsf{SL}_2(\mathbb{R})$, $\mathsf{GL}_2(\mathbb{R})$, $\mathrm{Mat}_2^+(\mathbb{R})$ and their higher dimensional counterparts. Let μ be a compactly supported probability measure on G.

Choose a matrix $g_0 \in G$ according to the probability law μ. Choose again a matrix $g_1 \in G$ according to μ and consider the product $g_1\, g_0$. Repeat this process n times to obtain the product

$$g_{n-1} \cdots g_1 g_0 \,,$$

and so on.

Probabilistic viewpoint

We study the multiplicative analogue of sums of i.i.d. random variables.

Consider a semigroup $G \subset \mathrm{Mat}_d(\mathbb{R})$. Common choices for G are $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{GL}_2(\mathbb{R})$, $\mathrm{Mat}_2^+(\mathbb{R})$ and their higher dimensional counterparts. Let μ be a compactly supported probability measure on G.

Choose a matrix $g_0 \in G$ according to the probability law μ. Choose again a matrix $g_1 \in G$ according to μ and consider the product $g_1\, g_0$. Repeat this process n times to obtain the product

$$g_{n-1} \cdots g_1 g_0 \,,$$

and so on.

The purpose of this work is the study of the almost sure limit

$$L_1(\mu) := \lim_{n\to\infty} \frac{1}{n} \log \|g_{n-1} \dots g_1 g_0\|.$$

Linear cocycles

Multiplicative processes of this, or a much more general type can be described in the framework of dynamical systems using the concept of linear cocycles.

Linear cocycles

Multiplicative processes of this, or a much more general type can be described in the framework of dynamical systems using the concept of linear cocycles.

A linear cocycle is a skew-product transformation

$$F\colon M \times \mathbb{R}^d \to M \times \mathbb{R}^d$$

defined by a pair (f, A), where $f\colon M \to M$ is called the base dynamics, $A\colon M \to \mathrm{Mat}_d(\mathbb{R})$ is the fiber map and

$$F(x, v) = (f(x), A(x)v)\,.$$

The n-th iterate of F is given by $F^n(x, v) = (f^n(x), A^n(x)v)$, where

$$A^n(x) := A(f^{n-1}(x)) \cdots A(f(x))A(x)\,.$$

Dynamical systems viewpoint

We use the concept of linear cocycle to describe the i.i.d. multiplicative process above.

Let μ be a probability measure in $G \subset \mathrm{Mat}_d(\mathbb{R})$ with compact support Σ and let M be the set of sequences $\Sigma^{\mathbb{Z}}$ endowed with the product measure $\nu = \mu^{\mathbb{Z}}$. Let $\sigma\colon M \to M$ be the Bernoulli shift over M and let $A\colon M \to G$ be the projection of the sequence to its zeroth coordinate.

Note that given $x = \{g_n\}_{n \in \mathbb{Z}} \in M$,

$$A^n(x) = A(\sigma^{n-1}(x)) \cdots A(\sigma(x))A(x) = g_{n-1} \cdots g_1 g_0$$

describes the product of i.i.d. matrices in G with probability law μ.

We call (σ, A) a *random locally constant linear cocycle*.

Lyapunov exponents

The central elements in the study of a linear cocycle are its Lyapunov exponents, defined as follows.

Theorem (Furstenberg-Kesten)

Let (M, f, ν) be an ergodic base dynamics and let $F \colon M \times \mathbb{R}^d \to M \times \mathbb{R}^d$, $F(x, v) = (f(x), A(x)v)$, where A is measurable and it satisfies the integrability condition $\log^+ \|A\| \in L^1(\nu)$. Then

$$L_1(F) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\|$$

exists for ν-almost every $x \in M$.

$L_1(F)$ is called the first Lyapunov exponent of F. Replacing in its definition the norm (i.e. the first singular value) by the second singular value, we obtain the second Lyapunov exponent $L_2(F)$ and so on.

Central problems on Lyapunov exponents

- Simplicity of the Lyapunov spectrum.

Central problems on Lyapunov exponents

- Simplicity of the Lyapunov spectrum.
- Availability of statistical properties such as large deviations type estimates and central limit theorems.

Central problems on Lyapunov exponents

- Simplicity of the Lyapunov spectrum.
- Availability of statistical properties such as large deviations type estimates and central limit theorems.
- Regularity with respect to the input data.

Central problems on Lyapunov exponents

- Simplicity of the Lyapunov spectrum.
- Availability of statistical properties such as large deviations type estimates and central limit theorems.
- Regularity with respect to the input data.

  A random, locally constant cocycle is determined by a probability measure μ on a semigroup $G \subset \mathrm{Mat}_d(\mathbb{R})$. An interesting problem is to investigate the regularity (or lack thereof) of the map

$$\mu \mapsto L_1(\mu)$$

  relative to various topologies on the space of probability measures, as well as choices of the semigroup G.

(Lack of) regularity for general cocycles

The regularity of Lyapunov exponents is in general a difficult problem, and it strongly depends on the topology in the space of cocycles.

(Lack of) regularity for general cocycles

The regularity of Lyapunov exponents is in general a difficult problem, and it strongly depends on the topology in the space of cocycles.

Theorem (Mañé-Bochi)

Let $f\colon M \to M$ be a homeomorphism and let μ be an ergodic and aperiodic probability measure on M (i.e. it gives zero measure to the set o periodic points). For any continuous *cocycle $A\colon M \to \mathrm{SL}_2(\mathbb{R})$, either A is uniformly hyperbolic or A is approximated in the C^0 topology by cocycles with $L_1(A) = 0$.*

(Lack of) regularity for general cocycles

The regularity of Lyapunov exponents is in general a difficult problem, and it strongly depends on the topology in the space of cocycles.

Theorem (Mañé-Bochi)

Let $f\colon M \to M$ be a homeomorphism and let μ be an ergodic and aperiodic probability measure on M (i.e. it gives zero measure to the set o periodic points). For any continuous *cocycle $A\colon M \to \mathrm{SL}_2(\mathbb{R})$, either A is uniformly hyperbolic or A is approximated in the C^0 topology by cocycles with $L_1(A) = 0$.*

Corollary

Under the conditions of the previous theorem, if $L_1(A) > 0$ then either A is uniformly hyperbolic (in which case the Lyapunov exponent is analytic near A by Ruelle) or else A is a point of discontinuity *of the Lyapunov exponents in the C^0 topology.*

Classical results for random cocycles in $\mathrm{GL}_d(\mathbb{R})$

The Lyapunov exponents of random, locally constant cocycles in the group $\mathrm{GL}_d(\mathbb{R})$ of invertible matrices do satisfy good regularity properties. We recall some of the more classical ones.

- Continuity in a generic setting: Furstenberg and Kifer.
- Hölder continuity in a generic setting: Le Page.
- Analyticity with respect to the probability weights for measures with *finite* support: Peres.
- Moreover, Ruelle proved the analyticity of the Lyapunov exponents for uniformly hyperbolic *general* cocycles.

Classical results for random cocycles in $\mathrm{GL}_d(\mathbb{R})$

The Lyapunov exponents of random, locally constant cocycles in the group $\mathrm{GL}_d(\mathbb{R})$ of invertible matrices do satisfy good regularity properties. We recall some of the more classical ones.

- Continuity in a generic setting: Furstenberg and Kifer.
- Hölder continuity in a generic setting: Le Page.
- Analyticity with respect to the probability weights for measures with *finite* support: Peres.
- Moreover, Ruelle proved the analyticity of the Lyapunov exponents for uniformly hyperbolic *general* cocycles.

Recent breakthrough (non-generic setting): Avila, Eskin, Viana.

Random *non* invertible cocycles

We now consider the setting of *singular* (i.e. non-invertible) random cocycles, whose Lyapunov exponents will show a strikingly different behavior, namely a *dichotomy* in the spirit of Mañé-Bochi.

Random *non* invertible cocycles

We now consider the setting of *singular* (i.e. non-invertible) random cocycles, whose Lyapunov exponents will show a strikingly different behavior, namely a *dichotomy* in the spirit of Mañé-Bochi.

We restrict to cocycles with values in the semigroup $\mathrm{Mat}_2^+(\mathbb{R})$ and measures with finite support. More precisely,

- Consider the alphabet $\mathcal{A} = \{1, \ldots, k\}$ and a partition $\mathcal{A} = \mathcal{A}_{\mathrm{sing}} \sqcup \mathcal{A}_{\mathrm{inv}}$ into two nonempty sets.
- Let $\underline{A} = (A_1, \ldots, A_k) \in \mathsf{Mat}_2^+(\mathbb{R})^k$ be such that $\mathrm{rank} A_i = 1$ if $i \in \mathcal{A}_{\mathrm{sing}}$ and $\det A_j > 0$ if $j \in \mathcal{A}_{\mathrm{inv}}$.
- Given a probability vector $p = (p_1, \ldots, p_k)$ with $p_i > 0$ for all i, the k-tuple $\underline{A} = (A_1, \ldots, A_k)$ determines the random linear cocycle $F \colon M \times \mathbb{R}^2 \to M \times \mathbb{R}^2$, $F(\omega, v) = (\sigma\omega, A(\omega)v) = (\sigma\omega, A_{\omega_0}\, v)$ where $M = \mathcal{A}^{\mathbb{Z}}$ is endowed with the product probability $\nu = p^{\mathbb{Z}}$.

Non-Invertible 2-D random linear cocycles

What do we know about non-invertible linear cocycles?

Non-Invertible 2-D random linear cocycles

What do we know about non-invertible linear cocycles?

- (*Duarte-Freijo*) When all the matrices admit a constant rank and $L_1(A) > L_2(A)$, then under generic conditions (irreducibility) the Lyapunov exponent is a Hölder continuous function.

Non-Invertible 2-D random linear cocycles

What do we know about non-invertible linear cocycles?

- (*Duarte-Freijo*) When all the matrices admit a constant rank and $L_1(A) > L_2(A)$, then under generic conditions (irreducibility) the Lyapunov exponent is a Hölder continuous function.
- "There is no hope to obtain any general regularity result for Lyapunov exponents in this more general setting."(of non-invertible cocycles) -*Avila, Eskin e Viana*.

Non-Invertible 2-D random linear cocycles

What do we know about non-invertible linear cocycles?

- (*Duarte-Freijo*) When all the matrices admit a constant rank and $L_1(A) > L_2(A)$, then under generic conditions (irreducibility) the Lyapunov exponent is a Hölder continuous function.
- "There is no hope to obtain any general regularity result for Lyapunov exponents in this more general setting."(of non-invertible cocycles) -*Avila, Eskin e Viana*.
- Both Duarte and Freijo, as well as Avila, Eskin and Viana comment independently about an example of a non-invertible cocycle with non constant rank (both 1 and 2) in which the Lyapunov exponent is discontinuous.

Non-Invertible 2-D random linear cocycles

Example

Consider the following matrices in $\mathrm{Mat}_2(\mathbb{R})$:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \tag{1}$$

and the family of cocycles $F_\alpha = ((\frac{1}{2}, \frac{1}{2}), (P, R_\alpha))$.

Non-Invertible 2-D random linear cocycles

Example

Consider the following matrices in $\mathrm{Mat}_2(\mathbb{R})$:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \tag{1}$$

and the family of cocycles $F_\alpha = ((\frac{1}{2}, \frac{1}{2}), (P, R_\alpha))$.

- If $\alpha \in \frac{\pi}{2\mathbb{Q}}$, there exists $n \in \mathbb{N}$ such that $PR_\alpha^n P = 0$ and $L_1(F_\alpha) = -\infty$.

Non-Invertible 2-D random linear cocycles

Example

Consider the following matrices in $\mathrm{Mat}_2(\mathbb{R})$:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \tag{1}$$

and the family of cocycles $F_\alpha = ((\frac{1}{2}, \frac{1}{2}), (P, R_\alpha))$.

- If $\alpha \in \frac{\pi}{2\mathbb{Q}}$, there exists $n \in \mathbb{N}$ such that $PR_\alpha^n P = 0$ and $L_1(F_\alpha) = -\infty$.
- $L_1(F_\alpha) = \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \log|\cos(j\alpha)|$.

Non-Invertible 2-D random linear cocycles

Example

Consider the following matrices in $\mathrm{Mat}_2(\mathbb{R})$:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \tag{1}$$

and the family of cocycles $F_\alpha = ((\frac{1}{2}, \frac{1}{2}), (P, R_\alpha))$.

- If $\alpha \in \frac{\pi}{2\mathbb{Q}}$, there exists $n \in \mathbb{N}$ such that $P R_\alpha^n P = 0$ and $L_1(F_\alpha) = -\infty$.
- $L_1(F_\alpha) = \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \log|\cos(j\alpha)|$.
- The set $\{\alpha \in \mathbb{R} \colon L_1(F_\alpha) > -\infty\}$ has full Lebesgue measure.

Non-Invertible 2-D random linear cocycles

Example

Consider the following matrices in $\mathrm{Mat}_2(\mathbb{R})$:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \tag{1}$$

and the family of cocycles $F_\alpha = ((\frac{1}{2}, \frac{1}{2}), (P, R_\alpha))$.

- If $\alpha \in \frac{\pi}{2\mathbb{Q}}$, there exists $n \in \mathbb{N}$ such that $PR_\alpha^n P = 0$ and $L_1(F_\alpha) = -\infty$.
- $L_1(F_\alpha) = \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \log|\cos(j\alpha)|$.
- The set $\{\alpha \in \mathbb{R} \colon L_1(F_\alpha) > -\infty\}$ has full Lebesgue measure.
- $L_1(F_\alpha)$ is discontinuous for almost every α.

Dominated splitting

Consider a linear cocycle $F\colon M \times \mathbb{R}^2 \to M \times \mathbb{R}^2$.

Definition

We say that this cocycle is *projectively uniformly hyperbolic* or that it admits a *dominated splitting* if there exists an F-invariant decomposition into one-dimensional subspaces $\mathbb{R}^2 = E_0(\omega) \oplus E_1(\omega)$ where the sub-bundles $M \ni \omega \mapsto E_i(\omega)$ are continuous functions and there exists $n \in \mathbb{N}$ such that $\|A^n|_{E_0}(\omega)\| < \|A^n|_{E_1}(\omega)\|$ for all $\omega \in M$.

Theorem (Ruelle)

Let A be a projectively uniformly hyperbolic cocycle. Then the map $B \mapsto L_1(B)$ is analytic in a neighborhood of A.

Hyperbolicity for random singular cocycles

A set $M \subset \mathbb{P}(\mathbb{R}^2)$ is called an *invariant multicone* for a random cocycle $\underline{A} = (A_1, \ldots, A_k) \in \mathrm{Mat}_2^+(\mathbb{R})^k$ if:

- M is open.
- $M \neq \mathbb{P}(\mathbb{R}^2)$ and $M \neq \emptyset$.
- $A_i M \Subset M$ for every $i \in \mathcal{A}$.

Hyperbolicity for random singular cocycles

A set $M \subset \mathbb{P}(\mathbb{R}^2)$ is called an *invariant multicone* for a random cocycle $\underline{A} = (A_1, \ldots, A_k) \in \mathrm{Mat}_2^+(\mathbb{R})^k$ if:

- M is open.
- $M \neq \mathbb{P}(\mathbb{R}^2)$ and $M \neq \emptyset$.
- $A_i M \Subset M$ for every $i \in \mathcal{A}$.

Theorem (Duarte, D-, Graxinha, Klein 2025)

Let $\underline{A} \in \mathrm{Mat}_2^+(\mathbb{R})^k$ be a random cocycle. The following are equivalent:

(i) *$\underline{A}$ is projectively uniformly hyperbolic.*

(ii) *$\underline{A}$ admits an invariant multicone.*

This extends a result of Avila, Bochi, Yoccoz from $SL_2(\mathbb{R})$-valued cocycles to our setting.

A Mañé-Bochi type dichotomy

The following result establishes a dichotomy in the spirit of Mañé-Bochi in a highly regular setting, that of locally constant linear cocycles over a Bernoulli shift in finite symbols. This is of course possible due to the existence of singular elements in the support of the measure.

Theorem (Duarte, D-, Graxinha, Klein 2025)

Consider the random cocycle determined by a tuple $\underline{A} = (A_1, \ldots, A_k) \in \mathrm{Mat}_2^+(\mathbb{R})^k$ with at least one singular and one invertible component, and a probability vector $p = (p_1, \ldots, p_k)$.

Then the cocycle $\underline{A}$ is either projectively uniformly hyperbolic or else there exists a sequence of tuples $\underline{A}_n \to \underline{A}$ having null words, so in particular $L_1(\underline{A}_n) = -\infty$ for all n.

Regularity dichotomy: analyticity vs. discontinuity

Corollary

Consider the random cocycle determined by a tuple $\underline{A} = (A_1, \dots, A_k) \in \mathrm{Mat}_2^+(\mathbb{R})^k$ *with at least one singular and one invertible component, and a probability vector* $p = (p_1, \dots, p_k)$.

If $L_1(\underline{A}) > -\infty$ *then in a neighborhood of* $\underline{A}$, *either the Lyapunov exponent is analytic or else it is discontinuous.*

Regularity dichotomy: analyticity vs. discontinuity

Corollary

Consider the random cocycle determined by a tuple $\underline{A} = (A_1, \ldots, A_k) \in \mathrm{Mat}_2^+(\mathbb{R})^k$ with at least one singular and one invertible component, and a probability vector $p = (p_1, \ldots, p_k)$.

If $L_1(\underline{A}) > -\infty$ then in a neighborhood of $\underline{A}$, either the Lyapunov exponent is analytic or else it is discontinuous.

Proposition

Lebesgue almost every cocycle $\underline{A} \in \mathrm{Mat}_2^+(\mathbb{R})^k$ with both singular and invertible components satisfies $L_1(\underline{A}) > -\infty$.

We thus conclude that the regularity of the Lyapunov exponent for random cocycles in $\mathrm{Mat}_2^+(\mathbb{R})^k$ with both singular and invertible components satisfies an *almost sure dichotomy* between *discontinuity* and *analyticity*.

Topological Point of View

Although from the point of view of measure theory we have a lot of discontinuities, we obtained a result in a different direction from the topological point of view...

Topological Point of View

Although from the point of view of measure theory we have a lot of discontinuities, we obtained a result in a different direction from the topological point of view...

Corollary

The set of continuity points of the Lyapunov exponent, which is $\mathcal{UH} \cup \{\underline{A}\colon L_1(\underline{A}) = -\infty\}$, *is a Baire residual subset.*

The identically singular case

We now consider the case when all components of the tuple $\underline{A} = (A_1, \ldots, A_k)$ are singular. We obtain the following.

Theorem (Duarte, D-, Graxinha, Klein 2025)

Let $\underline{A}$ be a random cocycle such that all of its components have rank 1. Then either $\underline{A}$ is projectively uniformly hyperbolic or else it admits null words, and in particular $L_1(\underline{A}) = -\infty$.

Corollary

At every random cocycle $\underline{A}$ such that all of its components have constant rank 1, the Lyapunov exponent is a continuous function. Moreover, it is locally analytic at Lebesgue almost every such cocycle.

Large deviations type estimates

Large deviations type estimates

After extending Furstenberg's theory to this setting, we were able to use a recent abstract large deviations result of Cai, Duarte, Klein and a relatively involved parameter elimination argument to derive the following.

Theorem (Duarte, D-, Graxinha, Klein 2025)

For Lebesgue almost every cocycle $\underline{A} \in \mathrm{Mat}_2^+(\mathbb{R})^k$ with both singular and invertible components, for every $\varepsilon > 0$ we have

$$\mathbb{P}\left\{ \left| \frac{1}{n} \log\|A^n\| - L_1(\underline{A}) \right| > \varepsilon \right\} \leq C\, e^{-c_0(\varepsilon)\, n^{1/3}}$$

where $C < \infty$, $c_0(\varepsilon) > 0$ is an explicit function of ε and $\mathbb{P} = p^{\mathbb{Z}}$ refers to the Bernoulli measure on the space of sequences of matrices.

A central limit theorem

Moreover, using an abstract central limit theorem of Gordin, Lifshitz and a relatively involved parameter elimination argument, we derive the following.

Theorem (Duarte, D-, Graxinha, Klein 2025)

For Lebesgue almost every cocycle $\underline{A} \in \mathrm{Mat}_2^+(\mathbb{R})^k$ with both singular and invertible components, there exists $\sigma > 0$ such that the following convergence in distribution to the normalized Gaussian holds:

$$\frac{\log\|A^n\| - n\, L_1(\underline{A})}{\sigma\,\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1)\,.$$

A summary of results

Together with previously available results, we obtain an almost complete picture regarding the regularity of the first Lyapunov exponent (R-LE) and the availability of large deviations type (LDT) estimates and of a central limit theorem (CLT) for two-dimensional, locally constant random cocycles in finite symbols.

For now, we understand the singular setting only when the invertible components have positive determinant. The general case is work in progress.

A summary of results

Together with previously available results, we obtain an almost complete picture regarding the regularity of the first Lyapunov exponent (R-LE) and the availability of large deviations type (LDT) estimates and of a central limit theorem (CLT) for two-dimensional, locally constant random cocycles in finite symbols.

For now, we understand the singular setting only when the invertible components have positive determinant. The general case is work in progress.

	R-LE	LDT	CLT
rank$=2$	(weak) Hölder	Yes	Yes
rank$=1$	analytic	Yes	Yes
rank$=1\&2$	discontinuous	Yes	Yes

Obrigado!

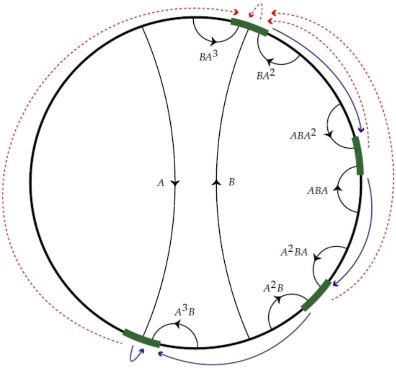

Figura: Multicone from ABY 2008.