

Integrable systems: from geometry to probability

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PLAN

1. **Integrable Systems**
2. **Algebraic and Geometric Aspects**
3. **Analytic and Probabilistic Aspects**

1. Integrable Systems

Classical Theory of Integrable Systems

An **Integrable System** is a Hamiltonian flow on a symplectic manifold (M^{2n}, ω) with global conserved quantities $H_1, \dots, H_n \in C^\infty(M)$ (H_1 being the Hamiltonian generating the flow) such that $\{H_i, H_j\} = 0$.

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Brief History

- ▶ two-body problem in celestial mechanics (Newton 1687)
- ▶ geodesics on ellipsoids, separation of variables in Hamilton–Jacobi equation (Jacobi 1837)
- ▶ integrability by quadratures of an integrable systems (Liouville 1838)
- ▶ harmonic oscillator on the unit sphere (Neumann 1859)
- ▶ Lagrange, Euler, and Kovalevskaya (1888) tops
- ▶ symmetries and integrals of motion (Noether 1915)
- ▶ global geometric version of Liouville theorem (Arnold 1963)

Modern Theory of Integrable Systems

Modern resurgence of interest (1960's→today)

Discovery of infinite-dimensional integrability (KdV, NLS, Toda, ...)

$$u_t = 6 u u_x - u_{xxx} \quad (KdV)$$

There are now infinitely many conserved Hamiltonians, generating a **hierarchy** of evolutionary PDEs, e.g.

$$u_{t_2} = 30 u^2 u_x - 20 u_x u_{xx} - 10 u u_{xxx} + u_{xxxxx}, \dots$$

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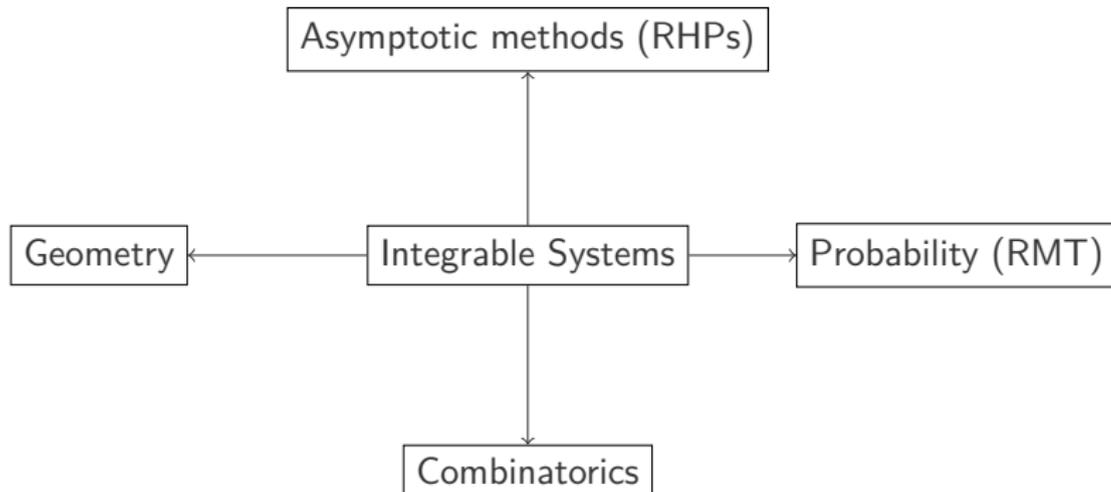
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- ▶ **solution schemes** and **analytic properties of solutions**: solitons, inverse scattering, Riemann–Hilbert problems, algebro-geometric solutions, asymptotics via nonlinear steepest descent method, ...
- ▶ **rich algebraic aspects**: infinite-Grassmannians and KP hierarchy, tau-functions as infinite-dimensional determinants,...
- ▶ **surprising connections**: algebraic geometry (Novikov conjecture on the Schottky problem, Witten conjecture, Gromov–Witten theory and Dubrovin–Frobenius manifolds), combinatorics, random matrix theory,...

Integrable Systems: a web of relationships



2. Algebraic and Geometric Aspects

Novikov's Conjecture on the Riemann–Schottky Problem

- ▶ Algebra–geometric functions solve integrable PDEs. Example:

$$u(x, t) = 2 \wp(x - ct \mid \Omega) - \frac{c}{6}$$

solves the KdV equation (**algebra-geometric integration**, extends to higher genus g / other integrable PDEs).

Novikov's Conjecture on the Riemann–Schottky Problem

- ▶ **Algebro–geometric functions solve integrable PDEs.** Example:

$$u(x, t) = 2 \wp(x - ct \mid \Omega) - \frac{c}{6}$$

solves the KdV equation (**algebro-geometric integration**, extends to higher genus g / other integrable PDEs).

- ▶ For a compact Riemann surface C of genus g , choose a symplectic basis of $H_1(C, \mathbb{Z})$. For a basis of holomorphic differentials $\omega_1, \dots, \omega_g$ normalized such that $\int_{a_i} \omega_j = \delta_{ij}$, the resulting **period matrix** is

$$\Omega_C = \left[\int_{b_i} \omega_j \right]_{i,j=1}^g \in \mathbb{H}_g \quad (\text{symmetric } g \times g \text{ matrix, } \operatorname{Im} \Omega_C > 0).$$

- ▶ **Riemann–Schottky problem:** characterize those matrices $\Omega \in \mathbb{H}_g$ that arise as period matrices of Riemann surfaces.
- ▶ **Novikov's conjecture (late '70s) – Shiota's theorem (1986):** Ω is a period matrix Ω_C of a curve C if and only if the associated **Riemann theta function** $\theta(z \mid \Omega)$ generates solutions of the **KP hierarchy**.

The Witten–Kontsevich Theorem

Physicists explored two approaches to Quantum Gravity in 2D (1980's)

1. *Path integral approach*: integrate over conformal structures on surfaces → intersection theory on the moduli space of curves.
2. *Discretization approach*: combinatorics of triangulations of surfaces → matrix models → KdV hierarchy.

E. Witten's physical insight: *quantum gravity should be unique*.

Postulating equality of correlators in the two approaches led to his

Witten's Conjecture, 1991 (proved by Kontsevich, 1992): Let

$$F(t_0, t_1, \dots) = \sum_{g,n} \frac{1}{n!} \sum_{d_1, \dots, d_n} \langle \tau_{d_1} \dots \tau_{d_n} \rangle_g \prod_{i=1}^n t_{d_i},$$

be the generating function of intersection numbers on $\overline{\mathcal{M}}_{g,n}$. Then $u = \partial^2 \log F_{t_0}$ satisfies the KdV hierarchy ($x = t_0$).

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Subsequent Developments: Deep interplay between *Gromov–Witten theory* and *integrable hierarchies of PDEs*. (Major contributors: Givental, Dubrovin, Kontsevich, Manin, Okounkov, Pandharipande,...)

My work in quantum KdV

Starting with the Witten–Kontsevich theorem, interactions between the algebraic aspects of integrable hierarchies of PDEs and enumerative geometry (e.g., Gromov–Witten theory and Dubrovin–Frobenius manifolds) have been intensively studied (both in Physics and Mathematics).

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My work concerns the **quantum KdV**, a commuting family of linear operators whose spectral problem is open. We obtained results concerning **explicit formulas** and relations to **quasimodular forms**

- ▶ Quantum KdV hierarchy and quasimodular forms (with J.-W. van Ittersum) Communications in Number Theory and Physics, 2024
- ▶ Quantum KdV hierarchy and shifted symmetric functions (with J.-W. van Ittersum) International Mathematics Research Notices, 2025

3. Analytic and Probabilistic Aspects

Random Matrix Theory and Integrable Systems

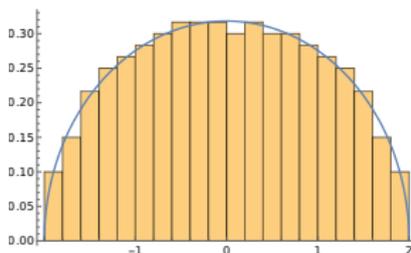
Gaussian Unitary Ensemble:

$M = (M_{ij})_{i,j=1}^n$, Hermitian ($M_{ij} = \overline{M_{ji}}$) with joint pdf of entries

$$\frac{1}{Z_n} \exp\left(-\frac{n}{2} \operatorname{tr}(M^2)\right) dM, \quad Z_n = 2^{n/2} (\pi/n)^{n^2/2}$$

(Equivalently, $X_{ij} \sim \mathcal{N}(0, 1) + i\mathcal{N}(0, 1)$ iid, $M = \frac{1}{2}(X + X^\dagger)/(2\sqrt{n})$.)

Typical eigenvalue distribution for $n \gg 1$:



Wigner semicircle law: $p(x) = \frac{1}{\pi} \sqrt{1 - \frac{x^2}{4}} \mathbf{1}_{[-2,2]}(x)$.

Random Matrix Theory and Integrable Systems

Largest eigenvalue $\lambda_{max} \sim 2$ as $n \rightarrow +\infty$.

Theorem (Tracy–Widom, 1993)

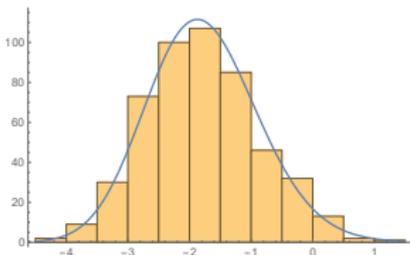
$$\lim_{n \rightarrow +\infty} \mathbb{P}((\lambda_{max} - 2)n^{2/3} \leq s) = F_{TW}(s)$$

where the *Tracy–Widom law* $F_{TW}(s)$ can be expressed as

$$F_{TW}(s) = \exp\left(-\int_s^{+\infty} (x - s) q(x)^2 dx\right)$$

where q satisfies the *Painlevé II equation*

$$q''(s) = sq(s) + 2q(s)^3, \quad q(s) \sim \text{Ai}(s) \text{ when } s \rightarrow +\infty.$$



Why do we care?

The Tracy–Widom law $F_{TW}(s)$ is *universal*

- ▶ it is independent (to a large extent) from the distribution law of the entries
- ▶ it arises beyond random matrix theory (e.g., random partitions, random permutations, trapped non-interacting fermions, tiling models, random interface growth models — all models of *integrable probability* [Borodin])

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Example: take uniformly sampled permutations π in S_n and consider the random variable

$\ell(\pi)$ = length of a longest increasing subsequence of π

(e.g., $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 6 & 3 & 5 \end{pmatrix} \Rightarrow \ell(\pi) = 3$)

Theorem (Baik–Deift–Johansson, 2000)

$$\lim_{n \rightarrow +\infty} \mathbb{P}((n^{-1/2}\ell_n - 2)n^{2/3} \leq s) = F_{TW}(s)$$

My work in integrable probability

Fact: The Tracy–Widom distribution can be written as a Fredholm determinant. This extends to many more interesting distributions in Exactly Solvable Systems.

Fredholm determinant \Rightarrow solutions to integrable PDEs \Rightarrow rich theory, many asymptotic tools.

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My work in integrable probability is about exploiting the connection to integrability to derive sharp asymptotics for the relevant probabilistic quantities of interest (KPZ, polynuclear growth models, ...):

- ▶ Airy kernel determinant solutions to the KdV equation and integro-differential Painlevé equations (with M. Cafasso & T. Claeys), *Communications in Mathematical Physics*, 2021
- ▶ Uniform tail asymptotics for Airy kernel determinant solutions to KdV and for the narrow wedge solution to KPZ (with C. Charlier and T. Claeys), *Journal of Functional Analysis*, 2022
- ▶ Integrable equations associated with the finite-temperature deformation of the discrete Bessel process (with M. Cafasso), *Journal of the London Mathematical Society*, 2023
- ▶ Jánossy densities and Darboux transformations for the Stark and cylindrical KdV equations (with T. Claeys, G. Glesner, & S. Tarricone), *Communications in Mathematical Physics*, 2024
- ▶ Multiplicative Averages of Plancherel Random Partitions: Elliptic Functions, Phase Transitions, and Applications (with M. Cafasso & M. Mucciconi), *arXiv:2601.05164*